# University of Liverpool Maths Club
# Ringing the Changes

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26 February 2005

It is not possible to ring tunes on English church bells, so instead we ring changes. Change ringing is about ringing a set of bells in different orders. These are called *rows* by bell ringers and *permutations* by mathematicians. Two examples of rows on eight bells are

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$$

and

$$1 \quad 3 \quad 5 \quad 7 \quad 2 \quad 4 \quad 6 \quad 8$$

Each row contains all eight bells, each exactly once.

**Question 1** *How many different permutations are there of three bells? Of four bells? Five? Eight? Twelve?*

**Question 2** *Suppose it takes two seconds to ring a complete row, no matter how many bells are involved. How long would it take to ring all the rows on three bells? Four? Five? Eight? (That has been done!) Twelve? (That hasn't!)*

To get from one row to the next, the rule is that any bell may only move at most one place. This means that the bells have to swap in pairs. We will work out how many different changes there are on any number of bells. To make it a little easier, we will count all the bells staying in the same place as a change too. We will write $C(n)$ for the number of changes on $n$ bells.

On one bell, there is one change: the bell has to stay in the same place! So $C(1) = 1$.

$$\begin{array}{c} 1 \\ | \\ 1 \end{array}$$

On two bells, there are two changes: either both bells stay in the same place, or they swap with each other. So $C(2) = 2$.

| | | |
|---|---|---|
| | 1 | 2 |
| 12 | \| | \| |
| | 1 | 2 |

| | | |
|---|---|---|
| | 1 | 2 |
| X | X | |
| | 2 | 1 |

On three bells there are three changes:

| | | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| 123 | \| | \| | \| |
| | 1 | 2 | 3 |

| | | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| 3 | X | | \| |
| | 2 | 1 | 3 |

| | | | |
|---|---|---|---|
| | 1 | 2 | 1 |
| 1 | \| | X | |
| | 1 | 3 | 2 |

and on four bells there are five changes.

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| 1234 | \| | \| | \| | \| |
| | 1 | 2 | 3 | 4 |

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| 34 | X | | \| | \| |
| | 2 | 1 | 3 | 4 |

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| 12 | \| | \| | X | |
| | 1 | 2 | 4 | 3 |

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| 14 | \| | X | | \| |
| | 1 | 3 | 2 | 4 |

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| X | X | | X | |
| | 2 | 1 | 4 | 3 |

**Question 3** *How many changes are there on five bells?*

We can put our results so far into a table.

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $C(n)$ | 1 | 2 | 3 | 5 | ? |

**Question 4** *It turns out that $C(n)$ is a familiar sequence of numbers. Do you know what it is? Can you prove it? (Hint: on $n$ bells, the first bell must either stay still or cross with the second bell. This should let you write down a formula relating $C(n)$ to $C(n-1)$ and $C(n-2)$.)*

We will now look at a way of ringing all the 24 possible permutations of 4 bells. To begin with, we will use the two changes 'X' and '14'.

Just as in the exercise we did during the talk, we start from rounds and alternate these two changes until we reach rounds again. This is what happens.

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| X | | | | |
| | 2 | 1 | 4 | 3 |
| 14 | | | | |
| | 2 | 4 | 1 | 3 |
| X | | | | |
| | 4 | 2 | 3 | 1 |
| 14 | | | | |
| | 4 | 3 | 2 | 1 |
| X | | | | |
| | 3 | 4 | 1 | 2 |
| 14 | | | | |
| | 3 | 1 | 4 | 2 |
| X | | | | |
| | 1 | 3 | 2 | 4 |
| 14 | | | | |
| | 1 | 2 | 3 | 4 |

If we stick to using those two changes, we can't produce any more than these eight rows! Something else is needed. What we will do is still ring these eight rows but, instead of going back to rounds at the end, we will replace the last '14' change with a '12' change.

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| X | | | | |
| | 2 | 1 | 4 | 3 |
| 14 | | | | |
| | 2 | 4 | 1 | 3 |
| X | | | | |
| | 4 | 2 | 3 | 1 |
| 14 | | | | |
| | 4 | 3 | 2 | 1 |
| X | | | | |
| | 3 | 4 | 1 | 2 |
| 14 | | | | |
| | 3 | 1 | 4 | 2 |
| X | | | | |
| | 1 | 3 | 2 | 4 |
| 12 | | | | |
| | 1 | 3 | 4 | 2 |

This is identical to last time except for the last row. Our plan is to repeat this whole pattern of changes until we get back to rounds.

**Question 5** *Look at the last row produced by the pattern of changes above. Which bell is in the same place as it started in? What has happened to the other three?*

**Question 6** *How many times will we need to repeat this sequence of changes before we arrive back at rounds? You should use the answer to the last question to help with this one.*

We will move the bottom row up to start a new column and repeat the same sequence of changes, starting from there.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | | 1 | 3 | 4 | 2 |
| X | 2 | 1 | 4 | 3 | X | 3 | 1 | 2 | 4 |
| 14 | 2 | 4 | 1 | 3 | 14 | 3 | 2 | 1 | 4 |
| X | 4 | 2 | 3 | 1 | X | 2 | 3 | 4 | 1 |
| 14 | 4 | 3 | 2 | 1 | 14 | 2 | 4 | 3 | 1 |
| X | 3 | 4 | 1 | 2 | X | 4 | 2 | 1 | 3 |
| 14 | 3 | 1 | 4 | 2 | 14 | 4 | 1 | 2 | 3 |
| X | 1 | 3 | 2 | 4 | X | 1 | 4 | 3 | 2 |
| 12 | | | | | 12 | 1 | 4 | 2 | 3 |

**Question 7** *Does the top row of the second column appear anywhere in the first column?*

**Question 8** *Remember that the first column contains all the rows that can be reached from rounds using the changes 'X' and '14'. Similarly, the second column contains all the rows that can be reached from its top row using those two changes (not counting the final row). What does that tell you about the rows in the first and second columns?*

Once again, we'll write the bottom row of the second column at the top of a new column, and apply the same sequence of changes.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | | 1 | 3 | 4 | 2 | | 1 | 4 | 2 | 3 |
| X | 2 | 1 | 4 | 3 | X | 3 | 1 | 2 | 4 | X | 4 | 1 | 3 | 2 |
| 14 | 2 | 4 | 1 | 3 | 14 | 3 | 2 | 1 | 4 | 14 | 4 | 3 | 1 | 2 |
| X | 4 | 2 | 3 | 1 | X | 2 | 3 | 4 | 1 | X | 3 | 4 | 2 | 1 |
| 14 | 4 | 3 | 2 | 1 | 14 | 2 | 4 | 3 | 1 | 14 | 3 | 2 | 4 | 1 |
| X | 3 | 4 | 1 | 2 | X | 4 | 2 | 1 | 3 | X | 2 | 3 | 1 | 4 |
| 14 | 3 | 1 | 4 | 2 | 14 | 4 | 1 | 2 | 3 | 14 | 2 | 1 | 3 | 4 |
| X | 1 | 3 | 2 | 4 | X | 1 | 4 | 3 | 2 | X | 1 | 2 | 4 | 3 |
| 12 | | | | | 12 | | | | | 12 | 1 | 2 | 3 | 4 |

We have arrived back at rounds!

**Question 9** *Does the row at the top of the third column appear in either of the first two columns? (You might be able to answer this without even looking at the numbers!)*

**Question 10** *What does that tell you about the rows in all three columns?*

**Question 11** *How many different rows have we rung?*

**Question 12** *How many permutations of the 4 bells have we not rung?*

If you would like to find out more about change ringing, see how it's done, or maybe even learn to ring, then please feel free to send me an email at `<mjbright@liv.ac.uk>`.