# Geometry and Arithmetic of Surfaces

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# 2

# The Hasse principle

In this chapter we introduce the notion of "local-global" techniques in the study of Diophantine equations: studying solutions of an equation over a global field $k$ (such as a number field) by looking at the solutions over the various local fields which are the completions of $k$.

## 2.1 Local solubility

In this book we are concerned with studying rational solutions to polynomial equations and, in particular, the question of whether a given set of polynomial equations has any rational solutions at all. If a polynomial equation defined over the rational numbers has no rational solutions, it can sometimes be very easy to prove this, as the following examples demonstrate.

**Example 2.1.1** The conic $x^2 + y^2 + z^2 = 0 \subset \mathbf{P}^2_{\mathbf{Q}}$ has no rational points, because it has no real points: the only solution to this equation in real numbers is $x = y = z = 0$.

**Example 2.1.2** The conic $x^2 + y^2 = 3z^2 \subset \mathbf{P}^2_{\mathbf{Q}}$ has no rational points. For suppose that $(x, y, z)$ were a solution. After multiplying them all by an appropriate constant, we could assume that $x, y, z$ were coprime integers. Then $x^2, y^2, z^2$ would each be congruent to 0 or 1 $\pmod 4$; looking at the equation shows that they would all have to be $0 \pmod 4$, and therefore $x, y, z$ would all be even, contradicting the assumption that they were coprime.

In both of these examples, we have proved that $X(\mathbf{Q}) = \emptyset$ by showing that $X(\mathbf{Q}_v) = \emptyset$ for some place $v$. In the first case it was $v = \infty$, the real place. In the second case we showed that $X(\mathbf{Z}/4\mathbf{Z})$ was empty, but we would like to think of this as showing the non-existence of solutions over $\mathbf{Q}_2$.

For any variety $X$ over a number field $k$, the condition that $X$ have points over every completion $k_v$ of $k$ is clearly a necessary condition for $X$ to have rational points over $k$. In order for such a condition to be useful in determining whether a given variety has rational points, we would like to have some procedure for determining when it holds. The first task in this section will be to show that deciding whether $X$ has points in every completion of $k$ can be achieved by a finite procedure.

Throughout this section, $k$ will denote a number field. We begin by looking at a single non-Archimedean completion $k_v$, where the indispensable tool is Hensel's Lemma. This comes in several guises; we will begin with the one-variable case. Two comprehensive resources for the various versions of Hensel's Lemma are Bourbaki (1998, Chapitre 3, §4) and Greenberg (1969, Chapter 5). We will denote the ring of integers of $k_v$ by $\mathfrak{o}_v$, and let $\pi$ be a uniformiser in $\mathfrak{o}_v$.

**Theorem 2.1.3** (Hensel's Lemma) *Let $f \in \mathfrak{o}_v[X]$ be a polynomial, and suppose that $x_0 \in \mathfrak{o}_v$ satisfies*

$$|f(x_0)|_v < (|f'(x_0)|_v)^2.$$

*Then there exists a unique $x \in \mathfrak{o}_v$ satisfying $|x - x_0|_v < |f'(x_0)|_v / |f(x_0)|_v$ and $f(x) = 0$.*

*Proof* See Cassels (1986, Chapter 4, Lemma 3.1). □

**Exercise 2.1.4** Show that an element $a \in \mathbf{Z}_2$ is a square if and only if the equation $X^2 - a$ has a solution modulo 8.

A straightforward corollary to the one-variable form of Hensel's Lemma is the following sufficient condition for the existence of a $k_v$-point on a projective hypersurface. Recall that a vector in $\mathfrak{o}_v^n$ is *primitive* if its entries do not all lie in the maximal ideal of $\mathfrak{o}_v$.

**Corollary 2.1.5** *Let $F \in \mathfrak{o}_v[X_0, \dots, X_n]$ be a homogeneous polynomial in $n$ variables. Suppose that $\mathbf{x_0} \in \mathfrak{o}_v^n \setminus \{\mathbf{0}\}$ is primitive and satisfies*

$$|F(\mathbf{x_0})|_v < (|(\partial F/\partial X_j)(\mathbf{x_0})|_v)^2$$

*for some $j$. Then there exists a non-zero $\mathbf{x} \in \mathfrak{o}_v^n$ satisfying $F(\mathbf{x}) = 0$.*

*Proof* Write $\mathbf{x_0} = (a_0, \dots, a_n)$, set $X_i = a_i$ for $i \neq j$, and apply Theorem 2.1.3 to the resulting polynomial in one variable $X_j$. □

For varieties defined by more than one polynomial, we need a more general version of Hensel's Lemma. A reasonably straightforward generalisation of the proof to matrices yields the following theorem.

**Theorem 2.1.6** *Let $f_1,\dots,f_r \in \mathfrak{o}_v[x_1,\dots,x_n]$ be polynomials, with $r \le n$, and denote by $\mathbf{J}$ the Jacobian matrix $\partial(f_1,\dots,f_r)/\partial(x_1,\dots,x_n)$. Given $\mathbf{x_0} \in \mathfrak{o}_v^n$, suppose that there is an $r \times r$ submatrix of $\mathbf{J}(\mathbf{x_0})$ of determinant $e$ satisfying*

$$\max_i \{|f_i(\mathbf{x_0})|_v\} < |e^2|.$$

*Then there exists $\mathbf{x} \in \mathfrak{o}_v^n$ satisfying $\max_i\{|\mathbf{x}_i - (\mathbf{x_0})_i|_v\} < |e|$ and, for all $i$, $f_i(\mathbf{x}) = 0$.*

*Proof* See Greenberg (1969, 5.21). □

A version of Theorem 2.1.6 with weaker hypotheses is given by Fisher (1997).

Armed with Hensel's Lemma, we can now show that it is indeed a finite procedure to decide whether a smooth variety has points over $k_v$. The most straightforward case is when the variety is a complete intersection in affine space.

**Proposition 2.1.7** *Let $f_1,\dots,f_r \in \mathfrak{o}_v[x_1,\dots,x_n]$ be polynomials defining a variety $V \subset \mathbf{A}^n_{k_v}$ that is smooth of dimension $n-r$, and such that $f_1,\dots,f_r$ generate the ideal $I(V,k_v)$. Then there is a finite procedure to decide whether there exists $\mathbf{x} \in \mathfrak{o}_v^n$ satisfying $f_i(\mathbf{x}) = 0$ for all $i$.*

*Proof* Let $\pi$ be a uniformiser in $\mathfrak{o}_v$. The algorithm proceeds as follows. Start with $m = 1$, and list all the solutions to $f_1 = \cdots = f_r = 0$ in $(\mathfrak{o}_v/(\pi^m))^n$. If there are none, then the equations are not soluble in $\mathfrak{o}_v$. If there is such a solution that, in addition, satisfies the condition of Theorem 2.1.6, then there exists a solution over $\mathfrak{o}_v$. Otherwise, increase $m$ and repeat.

It remains to show that this procedure terminates. Every solution modulo $\pi^{m+1}$ is obtained by lifting a solution modulo $\pi^m$. If the procedure were not to terminate, then it would instead construct (by lifting to $\mathfrak{o}_v$) an infinite sequence $\mathbf{x_1},\mathbf{x_2},\ldots \in \mathfrak{o}_v^n$ satisfying, for all $m$,

(i) $\mathbf{x_{m+1}} \equiv \mathbf{x_m} \pmod{\pi^m}$;
(ii) for all $i$, $|f_i(\mathbf{x_m})|_v < |\pi|_v^m$;
(iii) for every $r \times r$ submatrix $\mathbf{M}$ of $\mathbf{J}(\mathbf{x_m})$, we have $|\det \mathbf{M}|_v < |\pi|_v^{m/2}$.

These vectors form a Cauchy sequence in $\mathfrak{o}_v^n$, which therefore converges to some $\mathbf{x} \in \mathfrak{o}_v^n$. By continuity, we have $f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0$, and the rank of $\mathbf{J}(\mathbf{x})$ is strictly less that $r$. By the Jacobian criterion, this contradicts the assumption that $V$ is smooth of dimension $n-r$. □

To extend the proof to smooth varieties that are not complete intersections, we use the fact that a smooth variety admits an affine cover by complete intersections. As we will show, this follows from the Jacobian criterion for smoothness.

**Proposition 2.1.8** *Let $X \subset \mathbf{A}^n_{k_v}$ be a smooth affine variety, and let $f_1, \ldots, f_r \in \mathfrak{o}_v[x_1, \ldots, x_n]$ be generators for the ideal of $X$. Then there is a finite procedure to determine whether there exists $\mathbf{x} \in \mathfrak{o}_v^n$ satisfying $f_i(\mathbf{x}) = 0$ for all $i$.*

*Proof* This proof is closely related to that of Néron (1964, Proposition 20). Let $d$ be the dimension of $X$. Let $\mathbf{J}$ be the Jacobian matrix $\partial(f_1, \ldots, f_r)/\partial(x_1, \ldots, x_n)$. For every subset $S \subset [1, r]$ of size $n - d$, we consider the algebraic set $V_S$ defined by the set of polynomials $\mathscr{F}_S = \{f_i : i \in S\}$, and the corresponding Jacobian matrix $\mathbf{J}_S$ obtained by taking the corresponding rows of $\mathbf{J}$. Each $V_S$ contains $X$, so we have $\dim V_S \geq d$. Every $(n-d) \times (n-d)$ minor of $\mathbf{J}$ is also a minor of one of the $\mathbf{J}_S$; the Jacobian criterion for smoothness then shows that every point of $X$ is also a smooth point of dimension $d$ on some $V_S$. Let $\mathscr{S}$ denote the set of $S$ occurring in this way.

For $S \in \mathscr{S}$, the fact that $X$ and $V_S$ have a smooth point of dimension $d$ in common means that the variety $X$ is an irreducible component of $V_S$. Let $V'_S$ be the union of the remaining components. Extend $\mathscr{F}_S$ to a set of generators $\mathscr{F}_S \cup \mathscr{G}_S$ for the ideal $I(V'_S, k_v)$; then, for every point $x \in X \setminus (X \cap V'_S)$, we have $g(x) \neq 0$ for some $g \in \mathscr{G}_S$. Taking the union $\mathscr{G} = \bigcup_{S \in \mathscr{S}} \mathscr{G}_S$, we see that the sets $X \setminus \{g = 0\}$, for $g \in \mathscr{G}$, form a finite affine open cover of $X$.

We can scale the $g \in \mathscr{G}$ to have coefficients in $\mathfrak{o}_v$. Because at least one $g \in \mathscr{G}$ is non-zero at each point of $X$, the Nullstellensatz gives

$$(f_1, \ldots, f_r, \mathscr{G}) = (1) \subset k_v[x_1, \ldots, x_n]. \tag{2.1}$$

Working instead over $\mathfrak{o}_v$, the ideal generated by all these polynomials is not necessarily the unit ideal, but writing 1 in terms of the generators and clearing denominators shows that it must at least contain a constant:

$$\pi^s \in (f_1, \ldots, f_r, \mathscr{G}) \subset \mathfrak{o}_v[x_1, \ldots, x_n] \quad \text{for some } s \geq 0. \tag{2.2}$$

We deduce the following consequence: if $\mathbf{x} \in \mathfrak{o}_v^n$ is such that $f_i(\mathbf{x}) = 0$ for all $i = 1 \ldots, r$, then there exists $g \in \mathscr{G}$ such that $\pi^{s+1} \nmid g(\mathbf{x})$. Indeed, (2.2) gives

$$\sum_{i=1}^{r} a_i f_i + \sum_{g \in \mathscr{G}} b_g g = \pi^s$$

for suitable $a_i, b_g \in \mathfrak{o}_v[x_1, \ldots, x_n]$; evaluating at $\mathbf{x}$ shows that the $g(\mathbf{x})$ cannot all be divisible by $\pi^{s+1}$.

For any $S \in \mathscr{S}$ and any $g \in \mathscr{G}_S$, the equations

$$f_i(x_1, \ldots, x_n) = 0 \quad \text{for all } i \in S; \quad x_{n+1} g(x_1, \ldots, x_n) = \pi^s \tag{2.3}$$

in $n+1$ variables define a variety $Y_{S,g} \subset \mathbf{A}^{n+1}_{k_v}$ that is isomorphic to $X \setminus \{g=0\}$, and is therefore a smooth complete intersection of dimension $d$. Moreover, the solutions over $\mathfrak{o}_v$ to the equations (2.3) are in bijection with the $\mathbf{x} \in \mathfrak{o}_v^n$ satisfying $f_i(\mathbf{x}) = 0$ for all $i = 1,\dots,r$, and $\pi^{s+1} \nmid g(\mathbf{x})$. So every solution $\mathbf{x}$ to the original equations corresponds to a solution in $\mathfrak{o}_v$ to one of the finitely many sets of equations (2.3); and the existence of such solutions can be decided by Proposition 2.1.7. □

**Corollary 2.1.9** *Let $X \subset \mathbf{P}^n_{k_v}$ be a smooth projective variety. Then there is a finite procedure to determine whether $X(k_v)$ is empty.*

*Proof* We can scale the defining equations for $X$ so that they have coefficients in $\mathfrak{o}_v$. Any $k_v$-point on $X$ may be scaled to give a primitive $\mathfrak{o}_v$-point, which then corresponds to an $\mathfrak{o}_v$-point on one of the standard affine pieces of $X$. These may be tested for solubility by Proposition 2.1.8. □

We end this section by giving one of the most useful versions of Hensel's Lemma, which will play an important role in the following section.

**Proposition 2.1.10** *Let $X \subset \mathbf{A}^n_{k_v}$ be an affine variety of dimension d, and let $f_1,\dots,f_r \in \mathfrak{o}_v[x_1,\dots,x_n]$ be generators for the ideal of $X$. Denote by $\mathbf{F}$ the residue field of $\mathfrak{o}_v$. Let $\tilde{X} \subset \mathbf{A}^n_{\mathbf{F}}$ be the reduction of $X$ (see Definition 3.6.3) and suppose that the reductions $\tilde{f}_1,\dots,\tilde{f}_r \in \mathbf{F}[x_1,\dots,x_n]$ generate the ideal $I(\tilde{X},\mathbf{F})$. Suppose that $\tilde{\mathbf{x}} \in \tilde{X}(\mathbf{F})$ is a point at which $\tilde{X}$ is smooth of dimension d. Then there exists $\mathbf{x} \in \mathfrak{o}_v^n$ that lies in $X(k_v)$ and lifts $\tilde{\mathbf{x}}$.*

*Proof* We prove the case $d = n-r$, that is, when $X$ is a complete intersection. This case is a direct application of Theorem 2.1.6. Let $\mathbf{x_0} \in \mathfrak{o}_v^n$ be any lift of $\tilde{\mathbf{x}}$. The assumption that $\tilde{X}$ is smooth of dimension $n-r$ at $\tilde{\mathbf{x}}$ means that there is an $r \times r$ submatrix of the Jacobian matrix $\mathbf{J}(\mathbf{x_0})$ which reduces to an invertible matrix over $\mathbf{F}$, and so has determinant in $\mathfrak{o}_v^\times$. So the theorem applies and gives $\mathbf{x} \in \mathfrak{o}_v^n$ congruent to $\mathbf{x_0}$ modulo $v$, and satisfying $f_i(\mathbf{x}) = 0$ for all $i$.

In general, we reduce to the case of a complete intersection; what follows is a sketch of the argument. Let $\mathbf{J}$ be the Jacobian matrix $\partial(f_1,\dots,f_r)/\partial(x_1,\dots,x_n)$. The hypotheses show that there is a $(n-d)\times(n-d)$ submatrix of $\mathbf{J}(\tilde{\mathbf{x}})$ having non-zero determinant in $\mathbf{F}$; after renumbering, we may assume that the top-left $(n-d)\times(n-d)$ submatrix has this property. Denote by $R$ the polynomial ring $\mathfrak{o}_v[x_1,\dots,x_n]$. Let $I$ be the ideal $I(X,k_v)\cap R$, which by assumption is generated by $f_1,\dots,f_r$, and let $I' \subset I$ be the ideal in $R$ generated by $f_1,\dots,f_{n-d}$. The kernel of evaluation at the point $\tilde{\mathbf{x}}$ is a maximal ideal of $R$ whose images in $R/I$ and $R/I'$ we will denote $\mathfrak{m}$ and $\mathfrak{m}'$ respectively. Now the Jacobian criterion shows that the local rings $(R/I)_{\mathfrak{m}}$ and $(R/I')_{\mathfrak{m}'}$ are both regular, hence integral domains, and both of dimension $d+1$. Therefore the quotient map $(R/I')_{\mathfrak{m}'} \to (R/I)_{\mathfrak{m}}$ is an isomorphism, showing that image of the ideal $I$ in the ring $(R/I')_{\mathfrak{m}'}$ is the zero ideal. Since $I$ is finitely generated, it follows that there exists $g \in (R/I') \setminus \mathfrak{m}'$ satisfying $gI = 0$ in $R/I'$. In geometric terms, we have found $g \in R$ satisfying $g(\tilde{\mathbf{x}}) \neq 0$ and such that the open subset of $Z(\{f_1,\dots,f_{n-d}\},k)$ defined by $g \neq 0$ is an open subset of $X$. This open subset is isomorphic to the affine complete intersection $Y \subset \mathbf{A}_k^{n+1}$ defined by the equations $f_1(x_1,\dots,x_n) = \cdots = f_{n-d}(x_1,\dots,x_n) = 0$ and $x_{n+1}g(x_1,\dots,x_n) = 1$. Applying the proposition to $Y$ now gives the result. □

To complete this section on testing local solubility, we must also address Archimedean places. Solubility of a variety over the complex numbers $\mathbf{C}$ is determined by the Nullstellensatz: the affine variety defined by a set of polynomials is empty if and only if the polynomials generate the unit ideal. Over $\mathbf{R}$, there is an extensive theory of algorithmic algebraic geometry, and we quote the following result.

**Theorem 2.1.11** *Let $k$ be a number field and let $f_1,\dots,f_r \in k[x_1,\dots,x_n]$ be polynomials defining a variety $X \subset \mathbf{A}_k^n$. Let $v$ be a real place of $k$. There is a finite procedure to determine whether $X(k_v)$ is empty.*

*Proof* See Basu et al. (2006, Theorem 13.13). □

## 2.2 Everywhere local solubility

So far we have been studying how to tell whether a variety has points over a single completion $k_v$ of a number field $k$. However, for the local-global principle to be useful, we would like to be able to tell whether a given variety over $k$ has points in *all* completions of $k$.

**Definition 2.2.1** Let $k$ be a number field. The ring of *adèles* of $k$ is the restricted direct product $\mathbf{A}_k = \prod' k_v$ with respect to the rings of integers of the $k_v$. This is the subring of the direct product $\prod_v k_v$ consisting of those elements $(x_v)$ such that $x_v$ is an integer at all but finitely many places $v$. The set of *adelic points* of a variety $X$ over $k$ is the set $X(\mathbf{A}_k)$ of points of $X$ with coordinates in the adèles of $k$.

*Remark* 2.2.2 The notation $X(\mathbf{A}_k)$ makes sense: since $\mathbf{A}_k$ is a $k$-algebra, a polynomial in $k[x_1,\dots,x_n]$ can be evaluated at an $n$-tuple of elements of $\mathbf{A}_k$, so one can ask for the set of those $n$-tuples for which the evaluation is zero. It is easy to check that $X(\mathbf{A}_k)$ can be identified with those elements of the direct product $(P_v) \in \prod_v X(k_v)$ such that $P_v$ has coordinates which are integers in $k_v$, for all but finitely many places $v$.

If $X$ is projective, then we have the equality $X(\mathbf{A}_k) = \prod_v X(k_v)$. This is because any point of projective space over $k_v$ can be written with coordinates which are integers in $k_v$.

Using this notation and assuming that $X$ is projective, we see that $X(\mathbf{A}_k)$ is non-empty precisely when all of the $X(k_v)$ are non-empty, that is, when $X$ is everywhere locally soluble.

The key to showing that checking everywhere local solubility is a finite process is to show that, for almost all places, there is nothing to do. This follows from three important facts: given a smooth variety $X$ over $k$, its reduction at almost all places is also smooth; a smooth variety over a finite field has points over that field whenever the field is large enough; and smooth points over the residue field lift to points over the completion (Proposition 2.1.10). Let us address the first of these facts.

**Lemma 2.2.3** *Let $k$ be a number field with ring of integers $\mathfrak{o}$. Let $X \subset \mathbf{P}^n_k$ be a smooth, projective variety of dimension $d$ and let $f_1, \dots, f_r \in \mathfrak{o}[X_0, \dots, X_n]$ be polynomials generating the ideal $I(X,k)$. Then, for all primes $\mathfrak{p}$ of $\mathfrak{o}$ outside a finite computable set, the variety over $\mathbf{F}_\mathfrak{p} = \mathfrak{o}/\mathfrak{p}$ defined by the reductions of $f_1, \dots, f_r$ modulo $\mathfrak{p}$ is also smooth of dimension $d$.*

*Proof* Let $X$ have dimension $d$, and let $\mathscr{D} \subset \mathfrak{o}[X_0, \dots, X_n]$ be the set of all $(n-d) \times (n-d)$ minors of the Jacobian matrix $\partial(f_1, \dots, f_r)/\partial(X_0, \dots, X_n)$. The assumption that $X$ is smooth implies, by the projective Jacobian criterion, that $f_1, \dots, f_r$ together with all $D \in \mathscr{D}$ have no common zeros. By the Nullstellensatz, this means that the ideal they generate in $k[X_0, \dots, X_n]$ is the whole ring, and so one can compute polynomials $a_1, \dots, a_r$ and $(b_D)_{D \in \mathscr{D}}$, all in $k[X_0, \dots, X_n]$, satisfying

$$a_1 f_1 + \cdots + a_r f_r + \sum_{D \in \mathscr{D}} b_D D = 1.$$

Clearing denominators gives an expression

$$a'_1 f_1 + \cdots + a'_r f_r + \sum_{D \in \mathscr{D}} b'_D D = N \tag{2.4}$$

where $N \in \mathfrak{o}$ is an integer, and $a'_i = N a_i$ and $b'_D = N b_D$ all lie in $\mathfrak{o}[X_0, \dots, X_n]$. Now let $\mathfrak{p}$ be a prime of $\mathfrak{o}$ not dividing $N$. Reducing (2.4) modulo $\mathfrak{p}$ gives an identity

$$\tilde{a}'_1 \tilde{f}_1 + \cdots + \tilde{a}'_r \tilde{f}_r + \sum_{D \in \mathscr{D}} \tilde{b}'_D \tilde{D} = \tilde{N}$$

where $\tilde{N} \neq 0$ lies in $\mathbf{F}_\mathfrak{p}$. Therefore $\tilde{f}_1, \ldots, \tilde{f}_r$ and $\{\tilde{D} \colon D \in \mathscr{D}\}$ have no common zeros in $\mathbf{P}^n_{\mathbf{F}_\mathfrak{p}}$. Thus the variety $V(\tilde{f}_1, \ldots, \tilde{f}_r) \subset \mathbf{P}^n_{\mathbf{F}_\mathfrak{p}}$, which is known to have dimension $d$ by Lemma 3.6.8, satisfies the projective Jacobian criterion, so is smooth of dimension $d$. □

The next step is to show that, given a fixed variety $X$ over a number field $k$, we have $X(k_\mathfrak{p}) \neq \emptyset$ for all sufficiently large primes $\mathfrak{p}$ at which the reduction of $X$ is smooth. Given Proposition 2.1.10, it is sufficient to find a point on the reduction, and so we are led to the question of determining whether a variety over a finite field has any points. An elementary, but often useful, result in this direction is the following.

**Theorem 2.2.4** (Chevalley, Warning) *Let $\mathbf{F}$ be a finite field of characteristic $p$, and let $f_1, \ldots, f_r \in \mathbf{F}[x_1, \ldots, x_n]$ be polynomials. If $\sum_i \deg f_i < n$, then*

$$\#\{\mathbf{x} \in \mathbf{F}^n \mid f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0\}$$

*is divisible by $p$. In particular, if the $f_i$ are non-constant homogeneous polynomials, then they have a non-trivial common zero.*

*Proof* See Serre (1973, Chapter I, §2, Theorem 3). □

For more general results, we turn to the study of the number of points of a variety over a finite field, the subject of many deep results in arithmetic geometry. Our first such result is the Hasse–Weil bound for the number of points of a smooth curve over a finite field, proved by Hasse for elliptic curves and by Weil (1948, p. 70, Corollaire 3) for curves of any genus.

**Theorem 2.2.5** (Hasse, Weil) *Let $C$ be a smooth projective curve of genus $g$ over a finite field $\mathbf{F}$ of order $q$. Then the number of points of $C$ satisfies*

$$|\#C(\mathbf{F}) - (q+1)| \leq 2g\sqrt{q}.$$

Using this result inductively, Lang and Weil (1954) were able to prove a similar statement about higher-dimensional varieties, as follows.

**Theorem 2.2.6** (Lang, Weil) *There exists a constant $A(n,d,r)$ depending only on $n,d,r$ such that, for any finite field $\mathbf{F}$ of order $q$ and any projective variety $V \subset \mathbf{P}^n_\mathbf{F}$ of dimension $r$ and degree $d$, we have*

$$|\#V(\mathbf{F}) - q^r| \leq (d-1)(d-2)q^{r-\frac{1}{2}} + A(n,d,r)q^{r-1}.$$

A more sophisticated approach comes from the Weil conjectures, one of the crowning achievements of 20th-century mathematics. What follows is an extremely brief summary of how the proof of the Weil conjectures through étale cohomology can be applied to bound the number of points on a variety over a finite field. See Milne (1980) or Freitag and Kiehl (1988) for a thorough treatment.

Let $X_0$ be a smooth, projective, geometrically irreducible variety of dimension $d$ over a finite field $\mathbf{F}$ of $q$ elements. Let $\ell$ be any prime not dividing $q$ and let $\bar{X}_0$ denote the base change of $X_0$ to an algebraic closure of $\mathbf{F}$. The $\ell$-adic cohomology groups $\mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell)$ were defined, using étale cohomology, by Grothendieck and others; they are finite-dimensional vector spaces over $\mathbf{Q}_\ell$, and zero for $i > 2d$. It can be shown that the number of points of $X_0(\mathbf{F})$ is given by the Lefschetz trace formula:

$$\#X_0(\mathbf{F}) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(F^* \colon \mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell) \to \mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell)),$$

where $F^* \colon \mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell) \to \mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell)$ is the endomorphism on $\mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell)$ induced by the $q$-power Frobenius morphism on $\bar{X}_0$. Deligne (1974) proved that the eigenvalues of $F^*$ acting on $\mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell)$ are algebraic integers, all of whose conjugates have complex absolute value $q^{i/2}$. Since the trace of an endomorphism is simply the sum of the eigenvalues, this gives bounds on the traces of Frobenius in terms of the dimensions of the groups $\mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell)$. That $X_0$ is geometrically irreducible implies that $\mathrm{H}^{2d}(\bar{X}_0, \mathbf{Q}_\ell)$ and $\mathrm{H}^0(\bar{X}_0, \mathbf{Q}_\ell)$ both have dimension 1, giving

$$|\#X_0(\mathbf{F}) - (q^d + 1)| \leq \sum_{i=1}^{2d-1} q^{i/2} \dim \mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell). \tag{2.5}$$

If $X_0$ is the reduction of some variety $X$ over a number field, then the dimension of $\mathrm{H}^i(\bar{X}_0, \mathbf{Q}_\ell)$ is the same as the dimension of the complex cohomology group $\mathrm{H}^i(X(\mathbf{C}), \mathbf{C})$, that is, the $i$th Betti number $b_i(X(\mathbf{C}))$. We shall see in Chapter 9 how to compute the Betti numbers of some surfaces, in particular hypersurfaces in $\mathbf{P}^3$.

Putting this all together, we get the following statement.

**Proposition 2.2.7** *Let $k$ be a number field, and let $X \subset \mathbf{P}^n_k$ be a smooth, geometrically irreducible variety, given as the zero-set of a finite number of homogeneous polynomials. Suppose that the Betti numbers $b_i(X(\mathbf{C}))$ are known. Then there is a finite procedure to determine whether $X(\mathbf{A}_k)$ is empty.*

*Proof* If the given polynomials do not already generate the ideal $I(X,k)$, then there is an algorithm to compute new ones that do: see Cox et al. (2015, Section 4.2). Clearing denominators gives a set of generators with integer coefficients. So assume that $f_1, \ldots, f_r \in \mathfrak{o}[X_0, \ldots, X_n]$ generate $I(X,k)$. We further assume that $I(X,k)$ is not the unit ideal, since otherwise $X(\mathbf{A}_k)$ is certainly empty.

Using Lemma 2.2.3, we can find a finite set $S$ of primes of $k$ outside which the variety defined by $f_1, \dots, f_r$ is smooth. The inequality (2.5) gives a bound $M$ such that, for all finite places $\mathfrak{p}$ of $k$ outside $S$ and satisfying $\#(\mathfrak{o}/\mathfrak{p}) > M$, the equations $f_1 = \cdots = f_r = 0$ have a smooth solution in $\mathfrak{o}/\mathfrak{p}$ and so, by Proposition 2.1.10, $X(k_\mathfrak{p})$ is non-empty. At the remaining finite number of finite places, Corollary 2.1.9 shows that checking solubility of $X$ is a finite procedure. Solubility at any complex places is automatic. Solubility at real places can be checked by Theorem 2.1.11. □

## 2.3 The Hasse principle

We have seen that, for a smooth projective variety $X$ over a number field $k$, checking whether $X(\mathbf{A}_k)$ is non-empty is a finite process. For some families of varieties, this is enough to determine whether $X(k)$ is non-empty. The most famous example is the following theorem of Hasse and Minkowski, covering the case of quadratic forms.

**Theorem 2.3.1** (Hasse, Minkowski) *Let $k$ be a number field, and let $X \subset \mathbf{P}^n_k$ be defined by one quadratic form. If $X(\mathbf{A}_k)$ is non-empty, then $X(k)$ is non-empty.*

*Proof* See Serre (1973, Chapter 4, §3, Theorem 8). □

Because of this theorem, we say that quadratic forms satisfy the *Hasse principle*. We now define this principle more generally.

**Definition 2.3.2** Let $X$ be a variety over a number field $k$. We say that $X$ *satisfies the Hasse principle* or that the *Hasse principle holds for $X$* if the implication

$$X(k_v) \neq \emptyset \text{ for all places } v \text{ of } k \quad \Rightarrow \quad X(k) \neq \emptyset$$

holds. If this implication does not hold, then we say that $X$ is a *counterexample to the Hasse principle*.

In other words, a variety $X$ is a counterexample to the Hasse principle if $X$ has a point over every completion of $k$, but no point over $k$. If $X$ is projective, then $X$ satisfies the Hasse principle if and only if the implication $X(\mathbf{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$ holds.

It is usually more useful to talk not about whether an individual variety satisfies the Hasse principle, but rather whether a whole family of varieties satisfy the Hasse principle. For example, the Hasse–Minkowski theorem states that all quadrics satisfy the Hasse principle. In addition to quadrics, some other families of varieties are also known to satisfy the Hasse principle: for example, Severi–Brauer varieties (Châtelet, 1944); del Pezzo surfaces of degree at least 5 (due to various authors; see Várilly-Alvarado, 2013, Theorem 2.1); and varieties that are principal homogeneous spaces under simply connected algebraic groups (Kneser, Harder, Chernousov; see Platonov and Rapinchuk, 1994, Theorem 6.4 and Theorem 6.6).

Not all varieties satisfy the Hasse principle: here is an example, discovered independently by Lind (1940) and Reichardt (1942).

**Example 2.3.3** The curve of genus 1 defined by the equation

$$2Y^2 = X^4 - 17Z^4 \tag{2.6}$$

is a counterexample to the Hasse principle over $\mathbf{Q}$. In other words, this equation has solutions over $\mathbf{Q}_v$ for each place $v$, but has no rational solution.

*Remark* 2.3.4 The equation (2.6) is not homogeneous, so does not define a projective variety. There are two ways round this: either give the variable $Y$ weight 2, so that the equation defines a smooth variety in a weighted projective space; or take one affine piece, say by setting $Z$ equal to 1, form the projective closure of this affine curve, and then blow up to resolve the resulting singular point at infinity. The two procedures lead to isomorphic smooth, projective curves (since they have the same function field), of genus 1. To prove nonexistence of rational solutions, none of this matters, since it is immediately clear that any rational solution must have all of $X, Y, Z$ nonzero.

*Proof* Clearly there are real solutions. There are also solutions in $\mathbf{Q}_p$ for all $p \geq 3$ where the equation (2.6) has smooth reduction modulo $p$, since the Hasse bound (Theorem 2.2.5) says that any smooth curve of genus 1 over $\mathbf{F}_p$ has at least $p+1-2\sqrt{p} > 0$ points, and any of these lifts by Hensel's Lemma to a point over $\mathbf{Q}_p$. It only remains to check the finitely many primes of bad reduction (which are 2 and 17), and in each case a point is easily found.

We now show that there can be no rational solution to (2.6). If there were, then without loss of generality we could write it as $(X,Y,Z)$ with $X,Y,Z$ integers and $X,Z$ coprime. What primes may divide $Y$? If $q > 2$ is prime and $q \mid Y$, then reducing modulo $q$ gives $X^4 \equiv 17Z^4 \pmod{q}$ and so 17 is a square modulo $q$. By quadratic reciprocity, this means that $q$ is a square modulo 17.

Since 2 and $-1$ are also squares modulo 17, we deduce that $Y$ is a product of squares modulo 17 and thus $Y$ is a square modulo 17. We can therefore write $Y \equiv Y_0^2 \pmod{17}$. Substituting into (2.6), we get $2Y_0^4 \equiv X^4 \pmod{17}$

and hence that 2 is a fourth power modulo 17. But this is not true, and so there can be no rational solution. □

Most of the arguments in this proof are entirely local arguments: they involve making deductions about $X(\mathbf{Q}_v)$ for various places $v$. But there is one step which is not local, and that is the use of quadratic reciprocity. The theorem of quadratic reciprocity gives a link between behaviour at one prime and behaviour at another prime, and thus shows that the possible locations of our hypothetical rational solution in the various $X(\mathbf{Q}_v)$ are not independent of each other. We will see this technique repeated in the following examples.

**Example 2.3.5** (Birch and Swinnerton-Dyer, 1975) The surface $X$ defined by the equations

$$\begin{cases} uv = x^2 - 5y^2 \\ (u+v)(u+2v) = x^2 - 5z^2 \end{cases}$$

in $\mathbf{P}^4_{\mathbf{Q}}$ with coordinates $u,v,x,y,z$ is a counterexample to the Hasse principle.

*Proof* We begin by showing that $X$ has points everywhere locally. To do this, note that the points $[u : v : x : y : z] = [1 : 1 : 1 : 0 : \sqrt{-1}]$, $[10 : -10 : 5 : 5 : \sqrt{5}]$ and $[5 : 0 : 0 : 0 : \sqrt{-5}]$ all lie on $X$, and that, for any place $v \neq 2$, at least one of them is defined over $\mathbf{Q}_v$. As for $\mathbf{Q}_2$, the point $[-25 : 5 : 0 : 5 : 2\sqrt{-15}]$ lies in $X(\mathbf{Q}_2)$.

To show that $X$ has no rational points, we begin by supposing that there exists a rational solution $[u : v : x : y : z]$, where we may assume that $u,v$ are coprime integers (but the other coordinates need not be integers). Observe also that $(x,y) \neq (0,0)$.

Firstly we look at $X(\mathbf{Q}_5)$. The 5-adic valuations of $x^2$ and $5y^2$ are different, since one is even and the other is odd. Therefore both $x$ and $y$ are 5-adic integers, since otherwise $uv$ would not be a 5-adic integer. Now suppose that 5 divided $uv$; then 5 would divide $x$, and therefore 5 would divide $(u+v)(u+2v)$. But 5 can divide at most one of $u,v$, so we have a contradiction and deduce that 5 divides neither $u$ nor $v$. Similarly, 5 divides neither $(u+v)$ nor $(u+2v)$.

Now we use quadratic reciprocity, in the following guise: if an integer $n$ can be written as $n = x^2 - 5y^2$ for rational numbers $x$ and $y$, then every prime $p \equiv \pm 2 \pmod 5$ divides $n$ to an even power (see Lemma 16.2.7 below). We deduce that $uv$ (and hence $u$ and $v$ individually) are only divisible by such primes to even powers, and therefore that $u$ and $v$ are both congruent to $\pm 1 \pmod 5$. Similarly, both $(u+v)$ and $(u+2v)$ are congruent to $\pm 1 \pmod 5$. But these statements cannot all be true, and we conclude that no rational solution exists. □

The following lemma completes the proof.

**Lemma 2.3.6** *Let $n$ be an integer, and suppose that there exist rational numbers $x,y$ satisfying $n = x^2 - 5y^2$. If $p$ is a prime congruent to $\pm 2 \pmod 5$, then $v_p(n)$ is even.*

*Proof* Let $d$ be a common multiple of the denominators of $x$ and $y$. Multiplying through by $d^2$, which does not change the parity of $v_p(n)$ for any prime $p$, we may assume that $x$ and $y$ are integers. If now $e$ is the highest common factor of $x$ and $y$, then dividing by $e^2$ reduces to the case when $x$ and $y$ are coprime. We will show that, if $p$ is a prime satisfying $p \equiv \pm 2 \pmod 5$, then $p$ does not divide $n$.

Suppose that $p \neq 5$ is an odd prime dividing $n$. Reducing the equation modulo $p$ gives $x^2 - 5y^2 \equiv 0 \pmod p$. Since $x$ and $y$ are coprime, they are not both divisible by $p$, and so neither is. We get $5 \equiv (x/y)^2 \pmod p$, so 5 is a quadratic residue modulo $p$ and reciprocity shows $p \equiv \pm 1 \pmod 5$. $\square$

The surface $X$ in Example 2.3.5 is a del Pezzo surface of degree 4. We shall study del Pezzo surfaces in general in Chapter 7.

So far we have used the inclusion of the set of $k$-rational points of $X$ into the set of adelic points of $X$ to prove the non-existence of $k$-rational points. In fact, even if $X$ has $k$-rational points, we may be interested in understanding how accurately the set of rational points of $X$ is approximated by the set of adelic points of $X$, rather than just deciding whether it is empty. For instance, we can further ask whether $X$ satisfies weak approximation.

**Definition 2.3.7** A variety $X$ over a number field $k$ satisfies *weak approximation* if $X(k)$ is dense in $\prod_{v \in \Omega_k} X(k_v)$, with the product topology. Equivalently, given any open subsets $U_v \subset X(k_v)$ for finitely many places $v$ of $k$, there exists a point in $X(k)$ lying in each $U_v$ under the embedding $X(k) \subset X(k_v)$.

We conclude with one further example which, though not a counterexample to the Hasse principle, is a counterexample to weak approximation.

**Example 2.3.8** (Swinnerton-Dyer (1962)) The singular cubic surface $S$ defined by the equation

$$T(X^2 + Y^2) = (4Z - 7T)(Z^2 - 2T^2) \tag{2.7}$$

in $\mathbf{P}^3_{\mathbf{Q}}$ with homogeneous coordinates $X,Y,Z,T$ has real locus with two connected components. Rational points of $S$ are dense in one component; the other contains no rational points.

*Proof* To see the two connected components of the real locus, we look at the affine piece $T \neq 0$, given by the equation

$$x^2 + y^2 = (4z-7)(z^2-2) \tag{2.8}$$

in $\mathbf{A}^3_{\mathbf{Q}}$ with coordinates $x = \frac{X}{T}, y = \frac{Y}{T}, z = \frac{Z}{T}$. This is the surface of revolution about the $z$-axis of the elliptic curve

$$u^2 = (4z-7)(z^2-2). \tag{2.9}$$

The right-hand side of this equation is positive only for $|z| \leq \sqrt{2}$ and $z \geq 7/4$; these two ranges for $z$ determine the two connected components of the curve, and hence two connected components of the surface (2.8).

Firstly, we will show that rational points are dense in the component $z \geq 7/4$. The point $(x_0, y_0, z_0) = (1, 1, 2)$ lies in the surface. Consider the circle given by the intersection of the surface with the plane $z = z_0$. This is a plane conic with a rational point, and so has an isomorphism (given by projection away from the rational point) to $\mathbf{P}^1_{\mathbf{Q}}$. On $\mathbf{P}^1_{\mathbf{Q}}$ rational points are dense in the real points; we deduce that the same is true for the circle.

On the other hand, we can produce many more points to which this argument can be applied. The intersection of our surface with the plane $\{x = y\}$ is the elliptic curve $2v^2 = (4z-7)(z^2-2)$, and our point corresponds to the point $(1, 2)$ on this curve. That point lies in the same real component as the point at infinity, which is the identity element for the addition law on the elliptic curve. It turns out that our point has infinite order, and so its multiples are dense in that real component of the curve. We thus get a set of points of the affine surface (2.8) with $z$-coordinates dense in $\{z \geq 7/4\}$, and so a dense set of rational points on that connected component of the surface.

Secondly, we must prove that there are no rational solutions $[X : Y : Z : T]$ with $|Z/T| \leq \sqrt{2}$. We may assume that $Z, T$ are coprime integers and that $T > 0$. Multiplying the original equation (2.7) through by $T$ gives

$$T(7T - 4Z)(2T^2 - Z^2) = (TX)^2 + (TY)^2 \tag{2.10}$$

and, on this component, each of the left-hand terms $T$, $7T - 4Z$ and $2T^2 - Z^2$ is non-negative.

Quadratic reciprocity again appears in this proof in the guise of a well-known classical fact about quadratic forms: if $n$ is a positive integer which can be written as $n = a^2 + b^2$, with $a, b \in \mathbf{Q}$, then any prime congruent to 3 (mod 4) must divide $n$ to an even power. Applying this to (2.10) shows that, if $p \equiv 3 \pmod 4$, then the power of $p$ dividing the left-hand side must be even. We claim that, in fact, the power of $p$ dividing each of $T$, $7T - 4Z$ and

$2T^2 - Z^2$ must be even. To prove this, we look at their possible common factors and show that no such $p$ can divide more than one of them.

- Since $T$ and $Z$ are coprime, we have $(T, 7T-4Z) = (T, 4)$ so the only prime dividing both $T$ and $7T-4Z$ can be 2.
- $(T, 2T^2 - Z^2) = (T, Z^2) = 1$ so no prime can divide both $T$ and $2T^2 - Z^2$.
- Suppose that $p \equiv 3 \pmod 4$ divides $(7T-4Z, 2T^2-Z^2)$. Then $p$ also divides $(7T+4Z)(7T-4Z) - 16(2T^2-Z^2) = 17T^2$ and, since $p \neq 17$, then $p$ must divide $T$, which we have already seen is impossible.

Therefore none of $T$, $7T-4Z$, $2T^2-Z^2$ is congruent to $3 \pmod 4$. Since $Z$ and $T$ are coprime, if $T$ were even, then $Z$ would have to be odd, and therefore $2T^2 - Z^2 \equiv 3 \pmod 4$; whereas, if $T$ were odd, then $T$ would have to be congruent to $1 \pmod 4$ and therefore $7T - 4Z \equiv 3 \pmod 4$, giving a contradiction in either case. So there can be no rational solutions to (2.10), so none to (2.7) with $T \neq 0$ and $|Z/T| \leq \sqrt{2}$. □

# 4

# The Picard group

Given a set of polynomial equations defined over $\mathbf{Q}$, we aim to study their rational solutions by considering the geometry of the variety $X$ which they define. One geometric invariant which has a great effect on the arithmetic is the Picard group of $X$, and we will devote some time to the general definition of the Picard group and to understanding its structure for some specific surfaces.

## 4.1 Definition of the Picard group

One way to see the construction of the Picard group is to try to mimic the construction of the homology groups of a manifold. In that case, we form a free group of "cycles" and take the quotient by a subgroup of "boundaries". In the case of algebraic varieties, it is reasonable to replace the cycles by algebraic subvarieties. However, there is nothing immediately obvious to replace the boundaries, since a subvariety does not have a boundary. Many ways have been devised to solve this problem in arbitrary codimension, but in codimension one there is one which is particularly straightforward to define.

In what follows, $X$ will be a *smooth* irreducible variety over a field $k$. While the definitions will be valid for any field $k$, it may be easier at a first reading to imagine $k$ algebraically closed.

**Definition 4.1.1** A *prime divisor* on a smooth variety $X$ over a field $k$ is an irreducible closed subvariety $Z \subset X$ of codimension one, also defined over $k$. A *divisor* is a finite formal linear combination $D = \sum_i n_i Z_i$, $n_i \in \mathbf{Z}$, of prime divisors. The group of divisors on $X$, which is the free abelian group on the prime divisors, is denoted $\operatorname{Div} X$.

*Remark* 4.1.2 A prime divisor is not required to be nonsingular.

*Remark* 4.1.3 If $X$ is a variety over a field $k$ which is not algebraically closed, then a prime divisor does not have to be geometrically irreducible. For example, the 0-dimensional variety $\{\sqrt{2}, -\sqrt{2}\} \subset \mathbf{A}^1_{\mathbf{Q}}$, defined by the polynomial $x^2 - 2$, is irreducible as a variety over $\mathbf{Q}$, and is therefore a prime divisor on $\mathbf{A}^1_{\mathbf{Q}}$.

**Definition 4.1.4** A divisor $D$ is *effective* if it is a non-negative linear combination of prime divisors, i.e. if $D$ can be written as $D_1 + \cdots + D_r$, with $D_1, \ldots, D_r$ prime divisors.

**Definition 4.1.5** The *support* of a divisor $D$, written $\operatorname{supp} D$, is the closed subset of $X$ given by

$$\operatorname{supp}\left(\sum_i n_i Z_i\right) = \bigcup_{i \mid n_i \neq 0} Z_i.$$

The idea is that a divisor looks, at least locally, like the set of zeros and poles of a rational function. Let us make this precise. For any prime divisor $Z$ on $X$, we would like to define the valuation, or order of vanishing, of a function $f$ along $Z$. To do so, we first define the *local ring* $\mathscr{O}_{X,Z}$ at $Z$: this is the subring of the field of rational functions $\kappa(X)$ consisting of all those rational functions which are regular at some point of $Z$. The local ring contains a single maximal ideal $\mathfrak{m}$, which consists of those functions vanishing identically on $Z$. If a rational function $f \in \kappa(X)$ lies in the $d$th power of $\mathfrak{m}$ (that is $f \in \mathfrak{m}^d$), then we say that it vanishes to order at least $d$ along $Z$, if moreover $f \notin \mathfrak{m}^{d+1}$, then we say that $f$ has valuation $v_Z(f) = d$; if $1/f$ vanishes to order at least $d$, then we say that $f$ has a pole of order at least $d$ along $Z$, if moreover $1/f \notin \mathfrak{m}^{d+1}$, then we say that $v_Z(f) = -d$. If both $f$ and $1/f$ are in the local ring of $X$ at $Z$, then $v_Z(f) = 0$. If $X$ is a smooth variety and $Z$ is a prime divisor on $X$, general results in commutative algebra show that the only element of the local ring $\mathscr{O}_{X,Z}$ contained in all positive powers of the maximal ideal is the zero element and that $\mathscr{O}_{X,Z}$ is a discrete valuation ring, meaning that this definition gives a well-defined valuation $v_Z \colon \kappa(X)^\times \to \mathbf{Z}$.

Using valuations we can associate a divisor to any non-zero rational function on $X$.

**Definition 4.1.6** Let $f \in \kappa(X)^\times$ be a rational function on $X$. We define the *divisor of $f$* to be

$$\operatorname{div} f = (f) = \sum_Z v_Z(f) Z$$

where the sum is taken over all prime divisors $Z \subset X$.

*Remark* 4.1.7 This sum is finite – that is, $v_Z(f) = 0$ for all but finitely many prime divisors $Z$. To see this, suppose that $X$ is embedded in projective space $\mathbf{P}^N$. By definition, we can write $f$ as a quotient of two polynomials not vanishing on $X$; they are each zero only on a closed subset of codimension one in $X$, which is therefore the union of finitely many prime divisors.

**Proposition 4.1.8** *Let $X$ be a projective variety and let $f$ and $g$ be non-zero rational functions on $X$.*

(i) *If the divisor $(f)$ is 0, then $f$ is constant.*
(ii) *If the divisors $(f)$ and $(g)$ coincide, then there is a constant $a \in k^\times$ such that $f = ag$.*

*Proof* Statement (i) follows from Hartshorne (1977, Chapter I, Theorem 3.4(a)). Statement (ii) follows by applying (i) to the quotient $f/g$. □

**Definition 4.1.9** A divisor of the form $(f)$ for some $f \in \kappa(X)^\times$ is called a *principal divisor*. The subgroup of $\operatorname{Div} X$ consisting of the principal divisors is denoted by $\operatorname{Princ} X$.

**Definition 4.1.10** Two divisors $D, D' \in \operatorname{Div} X$ are *linearly equivalent*, written $D \sim D'$, if their difference $D - D'$ is principal.

**Example 4.1.11** Suppose that $D$ and $D'$ are two linearly equivalent, non-zero, effective divisors, whose supports have no component in common. Then, by definition, there is a function $f \in \kappa(X)^\times$ such that $(f) = D - D'$. Now the function $f$ defines a rational map from $X$ to $\mathbf{P}^1_k$, such that $f^{-1}(0) = D$ and $f^{-1}(\infty) = D'$. The other fibres of this rational map are all effective divisors which are also linearly equivalent to $D$, so give a “family” of effective divisors “moving” from $D$ to $D'$.

We can now define the Picard group of a smooth variety.

**Definition 4.1.12** Let $X$ be a smooth variety. The *Picard group* of $X$ is the quotient group

$$\operatorname{Pic} X = \frac{\operatorname{Div} X}{\operatorname{Princ} X}.$$

**Example 4.1.13** $\operatorname{Pic} \mathbf{A}^n = 0$ for any $n \geq 0$. To prove this, we must show that any irreducible subvariety of codimension one in $\mathbf{A}^n$ may be defined by a single polynomial. This reduces to the algebraic fact that, in a unique factorisation domain, any prime ideal of height one is principal. For a proof, see Eisenbud (1995, Corollary 10.6).

**Example 4.1.14** Let $D$ be a divisor on a smooth variety $X$ and let $P$ be a point of $X$. Then $D$ is linearly equivalent to a divisor $D'$ with $P \notin \operatorname{supp} D'$. For the local ring $\mathscr{O}_{X,P}$ is a unique factorisation domain, and so by using the same result as the previous example we can find a neighbourhood $U$ of $P$ and a rational function $f \neq 0$ such that $(f)$ and $D$ have the same restriction to $U$. Therefore $D' = D - (f)$ is a divisor linearly equivalent to $D$ and with support avoiding $U$ and hence $P$. Of course, the divisor $D'$ need not be effective, even if $D$ is effective.

**Example 4.1.15** Given a surface $X \subset \mathbf{P}^3$, a *plane section* is the divisor on $X$ defined by intersecting $X$ with a plane (and, if necessary, counting the components with the correct multiplicities). Any two plane sections of $X$ are linearly equivalent. For let $D_1$ and $D_2$ be the intersections of $X$ with the planes defined by linear forms $l_1$ and $l_2$ respectively. Then the quotient $l_1/l_2$ defines a rational function on $X$, with divisor $(l_1/l_2) = D_1 - D_2$.

More generally, let $X \subseteq \mathbf{P}^n$ be any projective variety. For the same reason, any two hyperplane sections of $X$ are linearly equivalent. We will often talk of "the" hyperplane section to mean the class in $\operatorname{Pic} X$ of a hyperplane section.

*Remark* 4.1.16 Bertini's Theorem (Hartshorne, 1977, Chapter II, Theorem 8.18) shows that, if $X$ is smooth and $k$ algebraically closed, then almost all hyperplane sections of $X$ are nonsingular. Generalisations of this result can give many consequences of the form "Any divisor $D$ is equivalent to a difference $A - B$ with $A, B$ effective and *special*", where *special* can mean, for example: smooth; avoiding a given finite set of points; transverse to a given finite set of subvarieties; and so on.

**Exercise 4.1.17** Let $Z$ be a prime divisor in a smooth variety $X$, and let $U$ denote the complement $X \smallsetminus Z$. Show that the sequence

$$\mathbf{Z} \to \operatorname{Pic} X \to \operatorname{Pic} U \to 0,$$

where the first map is $1 \mapsto Z$ and the second $D \mapsto D \cap U$, is exact.

**Exercise 4.1.18** Use the result of Exercise 4.1.17 to show that $\operatorname{Pic} \mathbf{P}^n \cong \mathbf{Z}$, for any positive integer $n$.

On a smooth curve, a divisor is a formal sum of zero-dimensional subvarieties, and it is straightforward to define the degree of a divisor.

**Definition 4.1.19** Let $X$ be a smooth curve over a field $k$, and let $Z$ be a prime divisor on $X$. The residue field $\kappa(Z)$ is a finite extension of $k$, and we define the *degree* of $Z$ to be $\deg(Z) = [\kappa(Z) : k]$. For a general divisor $D = \sum_i n_i Z_i$ on $X$, we define $\deg(D) = \sum_i n_i \deg(Z_i)$.

If $k$ is algebraically closed, then we have $\deg(Z_i) = 1$ for all $i$, so the formula reduces to $\deg(D) = \sum_i n_i$.

**Example 4.1.20** Take $X = \mathbf{A}^1_{\mathbf{Q}}$. For the prime divisor $Z_1 = \{0\}$, we have $\kappa(Z_1) = \mathbf{Q}$ and so $\deg(Z_1) = 1$. For the prime divisor $Z_2 = \{\sqrt{2}, -\sqrt{2}\}$ of Remark 4.1.3, we have $\kappa(Z_2) = \mathbf{Q}(\sqrt{2})$ and so $\deg(Z_2) = 2$.

**Proposition 4.1.21** *Let $X$ be a smooth, projective curve over a field $k$. For any function $f \in \kappa(X)$, the divisor $(f)$ has degree zero. Thus there is a well-defined degree map* $\deg\colon \operatorname{Pic} X \to \mathbf{Z}$.

*Proof* See Hartshorne (1977, II, Corollary 6.10) and either observe that the proof there works over arbitrary fields, or use Exercise 4.2.3 below to pass to an algebraic closure of $k$. □

*Remark* 4.1.22 We have been assuming throughout that $X$ is a smooth variety. For a general (not necessarily smooth) variety $X$, we can try to use the same definition, and it gives something called the *Weil divisor class group*. The Picard group in general is defined as the group of isomorphism classes of line bundles on $X$. If $X$ is normal, we can define a *Cartier divisor* to be a divisor $Z$ which is locally principal: that is, each point of $\operatorname{supp} Z$ has a neighbourhood in which $Z$ is principal. For an irreducible normal variety, the Picard group is isomorphic to the group of Cartier divisors modulo linear equivalence. This is equal to the Weil divisor class group if $X$ is locally factorial and, in particular, if $X$ is smooth. For a thorough treatment of these ideas, see Section II.6 of Hartshorne (1977).

## 4.2 Change of base field

Let $X$ be a smooth, geometrically irreducible variety over a field $k$, and let $\ell/k$ be a field extension. The relationship between $\operatorname{Pic} X$ and $\operatorname{Pic} X_\ell$ is subtle; in this section we gather some results about it. In the case of a Galois extension, we will be able to prove some of these results using cohomology in Chapter 14.

By definition, we have

$$\operatorname{Pic} X = \frac{\operatorname{Div} X}{\operatorname{Princ} X} = \frac{\text{Divisors on } X \text{ defined over } k}{\text{Divisors of functions defined over } k}$$

and

$$\operatorname{Pic} X_\ell = \frac{\operatorname{Div} X_\ell}{\operatorname{Princ} X_\ell} = \frac{\text{Divisors on } X \text{ defined over } \ell}{\text{Divisors of functions defined over } \ell}.$$

Given a prime divisor $Z$ on $X$, we can obtain a divisor on $X_\ell$ as follows. The base change $Z_\ell$ is a closed subset of codimension 1 in $X_\ell$, so decomposes as a finite union of irreducible components that are prime divisors: $Z_\ell = \bigcup_i Z_i$. Now let $t \in \mathscr{O}_{X,Z}$ be a uniformising element, and define $D_Z = \sum_i v_{Z_i}(t) Z_i$. The map sending each $Z$ to the corresponding $D_Z$ extends to a homomorphism from $\operatorname{Div} X$ to $\operatorname{Div} X_\ell$, which is injective since $Z$ is uniquely determined as the Zariski closure over $k$ of each $Z_i$. This allows us to consider $\operatorname{Div} X$ as a subgroup of $\operatorname{Div} X_\ell$.

**Example 4.2.1** Take $X = \mathbf{A}^1_{\mathbf{Q}}$ and let $Z$ be the prime divisor $\{\sqrt{2}, -\sqrt{2}\}$ of Remark 4.1.3. Take $\ell = \mathbf{Q}(\sqrt{2})$. The base change $Z_\ell$ decomposes into $Z_1 = \{\sqrt{2}\}$ and $Z_2 = \{\sqrt{2}\}$. The function $t = x^2 - 2$ is a uniformising element in $\mathscr{O}_{X,Z}$. On $X_\ell$ we have $v_{Z_1}(t) = v_{Z_2}(t) = 1$, and so the image of $Z$ in $\operatorname{Div} X_\ell$ is $Z_1 + Z_2$.

*Remark* 4.2.2 If $\ell/k$ is separable, then we have $v_{Z_i}(t) = 1$ for all $i$; we prove the important case, when $\ell/k$ is a finite separable extension. Write $\ell \cong k[x]/(f)$ and $K = \kappa(Z)$. By **?**, the ring $\mathscr{O}_{X,Z} \otimes_k \ell$ is isomorphic to the product of the rings $\mathscr{O}_{X_\ell, Z_i}$. Consider the ring

$$K[x]/(f) = K \otimes_k \ell = \mathscr{O}_{X,Z}/(t) \otimes_k \ell = \prod_i \mathscr{O}_{X_\ell, Z_i}/(t).$$

The polynomial $f$ is separable over $k$, hence also over $K$; so this ring is a product of field extensions of $K$. In particular, this means that $t$ is a uniformising element in each $\mathscr{O}_{X_\ell, Z_i}$.

**Exercise 4.2.3** If $X$ is a curve, show that the map $\operatorname{Div} X \to \operatorname{Div} X_\ell$ preserves degree.

**Exercise 4.2.4** Show that the map on divisor groups defined above is compatible with the natural map on function fields, that is, the diagram

$$\begin{array}{ccc} \kappa(X) & \xrightarrow{\text{div}} & \operatorname{Div} X \\ \downarrow & & \downarrow \\ \kappa(X_\ell) & \xrightarrow{\text{div}} & \operatorname{Div} X_\ell \end{array}$$

commutes.

It follows that there is an induced homomorphism $\operatorname{Pic} X \to \operatorname{Pic} X_\ell$.

Suppose that $\ell/k$ is a (possibly infinite) Galois extension. The Galois group $G = \operatorname{Gal}(\ell/k)$ acts on $\operatorname{Div} X_\ell$ and compatibly on $\kappa(X_\ell)$, inducing an action on $\operatorname{Pic} X_\ell$. The image of $i$ lies in the Galois-fixed subgroup $(\operatorname{Pic} X_\ell)^G$.

**Theorem 4.2.5** *Let $\ell/k$ be a Galois extension of fields, write $G = \mathrm{Gal}(\ell/k)$, and let $X$ be a variety defined over $k$. Let $i\colon \mathrm{Pic}\,X \to \mathrm{Pic}\,X_\ell$ be the natural homomorphism defined above.*

(i) $\mathrm{Div}\,X = (\mathrm{Div}\,X_\ell)^G$*, that is, a divisor on $X_\ell$ is defined over $k$ if and only if it is fixed by the Galois action.*
(ii) *If $X$ is a projective variety, then $i$ is injective.*
(iii) *If $X(k)$ is non-empty then $i$ gives an surjection from* $\mathrm{Pic}\,X$ *to* $(\mathrm{Pic}\,X_\ell)^G$.
(iv) *More generally, if $k$ is a number field and $X$ has points everywhere locally, that is, $X(k_v) \neq \emptyset$ for all places $v$ of $k$, then again $i\colon$ $\mathrm{Pic}\,X \to (\mathrm{Pic}\,X_\ell)^G$ is an surjection.*

*Proof* (i) A divisor in $\mathrm{Div}\,X_\ell$ is fixed by $G$ if and only if the coefficient of each prime divisor $W$ is equal to the coefficients of all its conjugates $\sigma W$, for $\sigma \in G$. Such a divisor can therefore be written as a sum

$$\sum_i n_i\Big(\sum_{\sigma\in G} \sigma W_i\Big)$$

for finitely many prime divisors $W_i$. By **??**, each closed subset $\bigcup_{\sigma\in G} \sigma W_i$ is defined over $k$ and is therefore a prime divisor $Z_i$ on $X$. Remark 4.2.2 shows that $\sum_{\sigma\in G} \sigma W_i$ is in fact the image of $Z_i$ in $\mathrm{Div}\,X_\ell$, completing the proof.
(ii) This comes down to saying that if a divisor $D$ is defined over $k$ and is the divisor of a function defined over $\ell$, then it is in fact the divisor of a function defined over $k$. This is an easy consequence of Hilbert's Theorem 90 (Proposition 14.7.13).
(iii) .

□

**Example 4.2.6** We give counterexamples to show that the various parts of the theorem can fail when the conditions are not satisfied.

If $X$ is not projective, then (ii) does not necessarily hold. Take any quadratic extension $\ell/\mathbf{Q}$ and let $X$ be $\mathbf{P}^1_{\mathbf{Q}}$ with a pair of points removed that are conjugate over $\ell$. Then the removed points constitute a prime divisor on $\mathbf{P}^1_{\mathbf{Q}}$, and Exercise 4.1.17 shows $\mathrm{Pic}\,X \cong \mathbf{Z}/2\mathbf{Z}$. However, the same exercise shows that $\mathrm{Pic}\,X_\ell$ is trivial.

If $X$ does not have points everywhere locally, then (iii) and (iv) do not have to hold. Let $X$ be a conic curve in $\mathbf{P}^2_{\mathbf{Q}}$ with no rational points, such as the curve $\{x^2 + y^2 + z^2 = 0\}$. The curve $X$ has many points over quadratic extensions, such as the point $P = [1 : 1 : i]$ defined over $\ell = \mathbf{Q}(i)$. The base change $X_\ell$ is isomorphic to $\mathbf{P}^1_\ell$, and therefore $P$ is linearly equivalent to its Galois conjugate – so the divisor class $[P]$ is fixed by $\mathrm{Gal}(\ell/\mathbf{Q})$. Since $[P]$ generates $\mathrm{Pic}\,X_\ell$, this means that $\mathrm{Gal}(\ell/\mathbf{Q})$ acts trivially on $\mathrm{Pic}\,X_\ell \cong \mathbf{Z}$. But there is no divisor of degree 1 on $X$, and in particular the class $[P]$ contains no divisor defined over $\mathbf{Q}$. The image of $\mathrm{Pic}\,X$ is of index 2 in $\mathrm{Pic}\,X_\ell$.

## 4.3 Intersection numbers

In this section, we let $X$ be a smooth *surface* over a field $k$. Given two curves in $X$, they will generally intersect in a finite number of points. The number of points is called their intersection number, and it gives us a very useful bilinear form on the Picard group.

**Definition 4.3.1** Let $C_1$ and $C_2$ be two curves on $X$, and let $P$ be a point over the algebraic closure $\bar{k}$ lying in both $C_1$ and $C_2$. We say that $C_1$ and $C_2$ *intersect transversely* at $P$ if, in the local ring $\mathscr{O}_{X,P}$, there are functions $f_1$, $f_2$ which generate the unique maximal ideal and are such that $(f_1) = C_1$ and $(f_2) = C_2$ on a neighbourhood of $P$. We say that $C_1$ and $C_2$ *intersect transversely* if they intersect transversely at all points of intersection.

This definition corresponds to the intuitive notion that the curves are nonsingular at $P$ and have distinct tangent directions.

**Example 4.3.2** In $\mathbf{A}^2_{\mathbf{Q}}$, the two coordinate axes intersect transversely at the origin $O$. They are defined by $\{x = 0\}$ and $\{y = 0\}$, and the maximal ideal of functions vanishing at $O$ is $(x, y)$.

**Example 4.3.3** Again in $\mathbf{A}^2_{\mathbf{Q}}$, consider the curves $\{x = 0\}$ and $\{y^2 = x^3\}$. They intersect at $O$, but the ideal generated by the functions $x$ and $y^2 - x^3$ is $(x, y^2)$ which is not the whole of the maximal ideal $(x, y)$. Therefore the curves do not intersect transversely.

It follows easily from the definition that the intersection of two curves that intersect transversely consists of isolated points (in the Zariski topology). Since any variety is quasi-compact in the Zariski topology, the two curves intersect in only finitely many points.

**Definition 4.3.4** Let $X$ be a smooth surface over a field $k$, and let $D$ and $D'$ be two prime divisors on $X$ which intersect transversely. We define the *intersection number* of $D$ and $D'$ to be

$$D \cdot D' = \#(D \cap D')$$

where the cardinality of the intersection $D \cap D'$ is taken over the algebraic closure of $k$.

**Theorem 4.3.5** *Let X be a smooth surface. The intersection number extends uniquely to a symmetric bilinear pairing* $\mathrm{Div}\,X \times \mathrm{Div}\,X \to \mathbf{Z}$ *which respects linear equivalence, and hence to a symmetric bilinear pairing* $\mathrm{Pic}\,X \times \mathrm{Pic}\,X \to \mathbf{Z}$.

*Proof* See Hartshorne (1977, Chapter V, Theorem 1.1). □

**Definition 4.3.6** Let $X$ be a smooth surface and $D$ a divisor on $X$. The *self-intersection number* of $D$ is the intersection number $D^2 = D \cdot D$.

A curve never intersects itself transversely, but we can make sense of the self-intersection as a number of intersection points by intersecting $D$ with a linearly equivalent divisor.

**Example 4.3.7** Any two distinct lines in $\mathbf{P}^2$ intersect transversely in precisely one point, and therefore have intersection number 1. Moreover, any line is linearly equivalent to any other line. We deduce that the self-intersection number of a line in $\mathbf{P}^2$ is 1.

**Example 4.3.8** Let $X \subset \mathbf{P}^n$ be a projective surface, and let $H$ be a hyperplane section of $X$. Then $H^2$ is the *degree* of $X$, defined to be the number of points of intersection of $X$ with any sufficiently general linear subspace of dimension $n-2$. To see this, use the fact that $H^2 = H_1 \cdot H_2$ where $H_1$ and $H_2$ are any two sufficiently general hyperplane sections of $X$.

**Exercise 4.3.9** Suppose that $X$ is a smooth hypersurface in $\mathbf{P}^3$ defined by a single equation of degree $d$. Show that the degree of $X$ is equal to $d$.

**Example 4.3.10** Let $X \subset \mathbf{P}^n$ be a projective surface, and let $C$ be an irreducible curve on $X$. Then $H \cdot C$ is the *degree* of $C$, defined to be the number of points of intersection of $C$ with a sufficiently general hyperplane.

**Exercise 4.3.11** Let $X$ be the projective quadric surface $xy = zw$, and let $U$ be the open subset defined by $w \neq 0$.

(i) Show that $U$ is isomorphic to $\mathbf{A}^2$, and deduce that $\mathrm{Pic}\,U = 0$.
(ii) Show that $X \setminus U$ consists of two straight lines. Using the exact sequence of Exercise 4.1.17, show that $\mathrm{Pic}\,X \cong \mathbf{Z}^2$, generated by the classes of these two straight lines.
(Hint: to show that the two lines are not equivalent, you may like to use intersection numbers.)

The intersection number defines a new equivalence relation on divisors on a surface.

**Definition 4.3.12** Let $X$ be a smooth surface. Two divisors $D$ and $D'$ on $X$ are said to be *numerically equivalent* if $D \cdot E = D' \cdot E$ for all divisors $E$ on $X$.

Given that intersection numbers respect linear equivalence, this gives an equivalence relation coarser than linear equivalence. The subgroup of classes in $\operatorname{Pic} X$ which are numerically equivalent to 0 is denoted by $\operatorname{Pic}^{\mathrm{n}} X$.

## 4.4 Structure of the Picard group over C

When $X$ is a smooth projective variety over the complex numbers $\mathbf{C}$, one can use methods from the theory of analytic varieties to deduce results about the Picard group of $X$. Here we mention briefly some useful facts arising from this.

There is an exact sequence of analytic sheaves on $X$ known as the exponential sequence, which gives rise to an exact sequence of cohomology groups:

$$\mathrm{H}^1(X(\mathbf{C}),\mathbf{Z}) \to \mathrm{H}^1(X,\mathscr{O}_X) \to \operatorname{Pic} X \to \mathrm{H}^2(X(\mathbf{C}),\mathbf{Z}).$$

We state several interesting facts about this sequence.

- Since $X$ is a smooth projective variety, $X(\mathbf{C})$ is a compact manifold. Its integral cohomology groups $\mathrm{H}^i(X(\mathbf{C}),\mathbf{Z})$ are therefore finitely generated abelian groups.
- The group $\mathrm{H}^1(X,\mathscr{O}_X)$ is a finite-dimensional complex vector space, and it turns out that $\mathrm{H}^1(X(\mathbf{C}),\mathbf{Z})$ is a lattice in this vector space. The image of $\mathrm{H}^1(X,\mathscr{O}_X)$ in $\operatorname{Pic} X$ is therefore a complex torus, and in fact is an Abelian variety. It is denoted $\operatorname{Pic}^0 X$, and lies inside the kernel $\operatorname{Pic}^{\mathrm{n}} X$ of the intersection pairing.
- The image of $\operatorname{Pic} X$ in $\mathrm{H}^2(X(\mathbf{C}),\mathbf{Z})$ is isomorphic to $\operatorname{Pic} X / \operatorname{Pic}^0 X$, and it is a finitely generated abelian group, called the *Néron–Severi group* of $X$.

For more background to these results, see Appendix B of Hartshorne (1977).

# 5

# Differentials and the canonical divisor

In this chapter we define differentials on a variety, and study some of their basic properties. The main purpose for us will be to define the canonical divisor class, which is important for classifying varieties and for defining embeddings of abstract varieties into projective space. Knowing something about the canonical class will also allow us to apply the Riemann–Roch theorem, which is a vital tool for studying the curves which lie in a surface. The emphasis in this chapter will be on understanding differentials in an explicit way, so that we can do calculations with them.

A standard and excellent reference for this material is Section II.8 of Hartshorne (1977).

## 5.1 Modules of differentials

Let $A$ be a commutative ring (with identity) and $B$ a commutative $A$-algebra. This means that there is a map $\iota\colon A \to B$, and we will often be careless in identifying elements of $A$ with their image in $B$. In particular, if $M$ is a $B$-module, then it is also an $A$-module, and we can write $am$ instead of $\iota(a)m$ for $a \in A$ and $m \in M$.

**Definition 5.1.1** An *A-derivation* of $B$ into a $B$-module $M$ is a map $d\colon B \to M$ such that:

(i) $d$ is additive, i.e. $d(b+b') = d(b) + d(b')$ for all $b, b' \in B$;
(ii) $d$ satisfies the Leibniz rule $d(bb') = bd(b') + b'd(b)$ for all $b, b' \in B$; and
(iii) $d(a) = 0$ for all $a \in A$.

It follows from properties (ii) and (iii) that $d$ is an $A$-linear map: that is, $d(ab) = ad(b)$ for all $a \in A$ and $b \in B$.

**Definition 5.1.2** The *module of relative differential forms* of $B$ over $A$ is a $B$-module $\Omega_{B/A}$, together with an $A$-derivation $d\colon B \to \Omega_{B/A}$, satisfying the following universal property: for any $B$-module $M$ and for any $A$-derivation $d'\colon B \to M$, there exists a unique $B$-module homomorphism $f\colon \Omega_{B/A} \to M$ such that $d' = f \circ d$.

**Exercise 5.1.3** Show that, if such an object exists, then it is unique up to unique isomorphism: that is, if two modules $\Omega_{B/A}$ and $\Omega'_{B/A}$ are both candidates satisfying the conditions of Definition 5.1.2, then there is a unique isomorphism between them. Hence we really can talk about "the" module $\Omega_{B/A}$.

If $\mathrm{Der}_A(B,M)$ denotes the set of all $A$-derivations from $B$ into $M$, then the universal property of $\Omega_{B/A}$ gives a natural bijection $\mathrm{Der}_A(B,M) \leftrightarrow \mathrm{Hom}_B(\Omega_{B/A},M)$.

**Proposition 5.1.4** *The module of relative differential forms $\Omega_{B/A}$ exists.*

*Proof* Let $F$ be the free $B$-module generated by a set of formal symbols $\{db : b \in B\}$. Define a submodule $R$ of $F$ generated by the following:

(i) $d(b+b') - db - db'$ for all $b,b' \in B$;
(ii) $d(bb') - bdb' - b'db$ for all $b,b' \in B$;
(iii) $da$ for all $a \in A$.

Now let $\Omega_{B/A}$ be the quotient $F/R$. Define $d\colon B \to \Omega_{B/A}$ by $b \mapsto db$; then, by construction, $d$ is a derivation. If $d' : B \to M$ is a derivation of $B$ into $M$, then define the homomorphism $f\colon \Omega_{B/A} \to M$ by $db \mapsto d'(b)$. This is well defined, since $d'$ is a derivation, and satisfies $d' = f \circ d$. It is also unique: since the symbols $db$ generate $\Omega_{B/A}$, any homomorphism $\Omega_{B/A} \to M$ is defined by where it sends the $db$, and so the requirement that $f(db) = d'b$ fixes $f$. □

**Exercise 5.1.5** Let $B = A[x_1,\dots,x_n]$ be a polynomial ring over $A$. Show that $\Omega_{B/A}$ is the free $B$-module of rank $n$ generated by the $dx_i$.

**Proposition 5.1.6** *Let $I$ be an ideal of $B$ and set $C = B/I$. Then there is a natural exact sequence of $C$-modules*

$$I/I^2 \xrightarrow{f} \Omega_{B/A} \otimes_B C \xrightarrow{g} \Omega_{C/A} \to 0,$$

*where $f$ sends $b \in I$ to $db \otimes 1$, and $g$ send $db \otimes c$ to $cd\bar{b}$, where $\bar{b}$ is the image of $b$ in $C$.*

*Proof* This proposition can be proved using more abstract methods, but we give an explicit proof.

Let us first check that $f$ is well defined. If $b \in I^2$, then we can write $b = xy$ with $x, y \in I$. Then $db = xdy + ydx$, and so

$$db \otimes 1 = x(dy \otimes 1) + y(dx \otimes 1) = dy \otimes \bar{x} + dx \otimes \bar{y} = 0.$$

Moreover, $f$ is a $C$-linear map. If $b \in I$ and $c \in C$, lift $c$ to an element $b' \in B$; then the action of $c$ on $I/I^2$ takes $b$ to the class of $b'b$. Now

$$d(b'b) \otimes 1 = (b'db + bdb') \otimes 1 = (db \otimes c) + (db' \otimes b) = c(db \otimes 1)$$

proving that $f$ is indeed $C$-linear.

To show exactness, we will look explicitly at systems of generators and relations for the modules $\Omega_{B/A} \otimes_B C$ and $\Omega_{C/A}$, as described in the proof of Proposition 5.1.4. It is a general fact about tensor products that, if a $B$-module $M$ is described by generators and relations, then $M \otimes_B C$ is described by the same generators, and the same relations but with coefficients pushed into $C$. So $\Omega_{B/A} \otimes_B C$ is the $C$-module generated by symbols $db$ for all $b \in B$, where we identify $db$ with $db \otimes 1$. The relations for additivity and for vanishing on $A$ are unchanged from $\Omega_{B/A}$, but for the Leibniz rule we should take relations $d(bb') - \bar{b}db' - \bar{b}'db$, where $\bar{b}$ denotes the image of $b$ in $C$.

Now consider the module $\Omega_{C/A}$. This is generated by symbols $dc$ for all $c \in C$, but (since $B \to C$ is surjective) we can equally well generate it by $db$ for all $b \in B$, as long as we add relations $db - db'$ whenever $b - b' \in I$. Having done this, it makes no difference whether we take additivity relations $d(c + c') - dc - dc'$ for $c \in C$, or $d(b + b') - db - db'$ for all $b \in B$. A similar statement holds for the Leibniz rule relations.

To summarise, we have presentations for both $\Omega_{B/A} \otimes_B C$ and $\Omega_{C/A}$ as $C$-modules. They both have the same set of generators $\{db\}$, and their set of relations differ only in that $\Omega_{C/A}$ has the additional relations $db - db'$ whenever $b - b' \in I$; equivalently (using the additivity relation) the additional relations can be $db$ for all $b \in I$. Therefore the natural map $\Omega_{B/A} \otimes C \to \Omega_{C/A}$ is surjective, and its kernel is precisely the submodule generated by the $db \otimes 1$ for all $b \in I$, as stated. □

**Example 5.1.7** If $A = k$ is a field, $B = k[x, y]$, and $C = k[x, y]/(x^2 + y^2 - 1)$, then $\Omega_{C/A}$ is the $C$-module generated by $dx$ and $dy$ with relation $2xdx + 2ydy = 0$.

**Example 5.1.8** If $A = k$ is a field, and $B = k[\varepsilon]/(\varepsilon^2)$, then $\Omega_{B/A}$ is the $B$-module generated by $d\varepsilon$ and the relation $2\varepsilon d\varepsilon = 0$.

There are many more interesting exact sequences describing for instance the behavior of modules of relative differential forms under tensor products, and the relation of the modules $\Omega_{B/A}$, $\Omega_{C/A}$, $\Omega_{C/B}$ for any $B$-algebra $C$. For more information, see Hartshorne (1977, Section II.8). Here we will only present what we need for our purposes.

**Proposition 5.1.9** *If $B$ is an integral domain with fraction field $K$, then $\Omega_{K/A} \cong \Omega_{B/A} \otimes_B K$.*

## 5.2 Differentials on varieties

For this section, let $k$ be a field. If $X$ is a smooth, irreducible variety over $k$, we will define the regular differentials on $X$ as a subset of the vector space of differentials $\Omega_{\kappa(X)/k}$.

**Definition 5.2.1** Let $X$ be a smooth, irreducible variety over a field $k$. A differential $\omega \in \Omega_{\kappa(X)/k}$ is *regular* at a point $P \in X$ if there exists an affine open neighbourhood $U \subset X$ of $P$, with coordinate ring $B$, such that the $B$-submodule $\Omega_{B/k}$ of $\Omega_{\kappa(X)/k}$ contains $\omega$.

**Example 5.2.2** Consider $\mathbf{P}^1_k(x,y)$ with function field $k(t)$ for $t = x/y$, and set $\omega = dt$. Then the affine part $y \neq 0$ can be identified with $\mathbf{A}^1(t)$, which has coordinate ring $k[t]$. Since the $k[t]$-module $\Omega_{k[t]/k}$ is generated by $dt$, we find that $\omega$ is regular at every point of $\mathbf{A}^1$.

In the following definition, when we require an object to be regular at every point of a subset, we really mean *every* point, whatever field it may be defined over. It is certainly not enough to check the condition only for points of $X(k)$.

**Definition 5.2.3** For every open subset $U \subset X$ let $\mathscr{O}_X(U)$ denote the subring of $\kappa(X)$ of functions that are regular at every point of $U$, and let $\Omega_X(U)$ denote the $\mathscr{O}_X(U)$-submodule of $\Omega_{\kappa(X)/k}$ consisting of differentials $\omega$ that are regular at every point of $U$.

*Remark* 5.2.4 We have in fact just defined two sheaves: the structure sheaf $\mathscr{O}_X$ and the sheaf of differentials $\Omega_X$ on the variety $X$.

**Proposition 5.2.5** *If $U$ is an affine subvariety of $X$ with coordinate ring $B$, then we have $\Omega_X(U) = \Omega_U(U) = \Omega_{B/k}$.*

*Proof* Since fhe function fields $\kappa(X)$ and $\kappa(U)$ are equal, this follows immediately from Definition 5.2.1. □

Given a variety $X$ over $k$, are there differentials on $X$ which are regular everywhere? The following example shows that this is not the case for the projective line.

**Example 5.2.6** Consider $\mathbf{P}^1_k(x,y)$ with function field $K = k(t) = k(s)$ for $t = x/y$ and $s = t^{-1}$. It follows from Exercise 5.1.5 and Proposition 5.1.9 that $\Omega_{K/k}$ is the 1-dimensional vector space over $K$ generated by $dt$ (or, equally well, by $ds$). The regular differentials on $\mathbf{A}^1(t)$ are in the $k[t]$-module generated by $dt$, while those on $\mathbf{A}^1(s)$ are in the $k[s]$-module generated by $ds = d(t^{-1}) = -t^{-2}dt$. The intersection in $\Omega_{K/k}$ is 0, so $\Omega_{\mathbf{P}^1}(\mathbf{P}^1) = 0$, i.e., there are no nonzero differentials that are regular on $\mathbf{P}^1$.

**Exercise 5.2.7** Consider the differential $\omega = dx/y$ on the affine curve $C$ in $\mathbf{A}^2_k(x,y)$ given by $y^2 = f(x)$ for some polynomial $f$ with no repeated roots. Show that $\omega$ is regular at every point of $C$. Show that this is consistent with Proposition 5.2.5. Show that if $f$ has degree 3, then $\omega$ is in fact regular on the entire projective closure of $C$ in $\mathbf{P}^3$.

**Exercise 5.2.8** (*) Show that if $X$ is a hypersurface in $\mathbf{P}^n$ for $n \geq 3$, then $\Omega_X(X) = 0$.

**Exercise 5.2.9** (*) Show that if $X$ is a complete intersection in $\mathbf{P}^n$ of dimension at least 2, then $\Gamma(\Omega_X) = 0$.

When studying the rational points on a variety $X$ over a number field, it is often tempting to try to map $X$ to another variety $Y$ on which we can control the set of points more easily. Every rational point on $X$ would then map to a rational point of $Y$, so all we would need to check is which rational points of $Y$ lift to rational points on $X$, a process that is particularly easy when $Y$ does not contain any rational points. Given $X$, the following proposition gives restrictions on $Y$ (to the extend that it may show this approach is useless for $X$).

**Proposition 5.2.10** *If $f\colon X \to Y$ is a surjective morphism of smooth irreducible varieties over a field $k$, and $f$ is generically smooth (which is automatic in characteristic 0), then then the induced map $f^*\colon \Omega_Y(Y) \to \Omega_X(X)$ is injective.*

*Proof* Since $f$ is generically smooth, there is an open subset $U \subset X$ such that $f\colon U \to Y$ is smooth (cf. Hartshorne, 1977, Lemma III.10.5). This is equivalent to saying that for any point $x \in U$, and $y = f(x)$, the induced map $T_x \to T_y$ on Zariski tangent spaces is surjective, or equivalently, the map $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is injective, where $\mathfrak{m}_x$ and $\mathfrak{m}_y$ denote the maximal ideals of the local rings at $x$ and $y$ respectively (see Hartshorne, 1977, Proposition III.10.4). Now take any nonzero differential $\omega \in \Gamma(\Omega_Y)$. Since $f(U)$ is dense in $Y$ and $\omega$ can not vanish on an open subset, there is a $y \in f(U)$ such that $\omega$ does not vanish at $y$. Take any $x \in U$ such that $f(x) = y$. Then by the above the map $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is injective. These $k$-vector spaces are the stalks at $y$ and $x$ of the sheaves $\Omega_{Y/k}$

and $\Omega_{X/k}$ respectively. Since $\omega$ does not vanish at $y$, its image in the stalk at $y$ is nonzero, and by injectivity, so is its image in the stalk at $x$, and therefore the image of $\omega$ in $\Omega_X(X)$ is nonzero. $\square$

**Corollary 5.2.11** *Let $X$ be a smooth variety over a field $k$. If $\Omega_X(X) = 0$, and* $\operatorname{char} k = 0$*, then there is no surjective morphism from $X$ to a nonsingular curve of positive genus or to an abelian variety of positive dimension.*

*Proof* Suppose $f\colon X \to Y$ is a surjective morphism for some $Y$. Since the characteristic is zero, the morphism $f$ is generically smooth, so Proposition 5.2.10 tells us that $\Omega_Y(Y) = 0$. This prevents $Y$ from being a curve of genus $g > 0$, or an abelian variety of dimension $g > 0$, both of which would satisfy $\dim_k \Omega_Y(Y) = g$. $\square$

Exercise 5.2.9 and Corollary 5.2.11 show that if $X$ is a complete intersection of dimension at least 2, then there is no hope for a morphism from $X$ to a curve of positive genus or an abelian variety of positive dimension.

We finish this section with some results describing the dimension of $\Omega_{K/k}$ when $K/k$ is a field extension.

**Definition 5.2.12** Let $K$ be a field extension of $k$. Then $K$ is *separably generated* over $k$ if there exists a transcendence basis $\{t_i\}$ for $K/k$ such that $K$ is a separable algebraic extension of $k(\{t_i\})$.

**Proposition 5.2.13** *Let $K$ be a finitely generated extension field of a field $k$. Then* $\dim_K \Omega_{K/k} \geq \operatorname{tr.deg.} K/k$*, with equality if and only if $K$ is separably generated over $k$.*

*Proof* See [somewhere]. $\square$

**Corollary 5.2.14** *Let $K = \kappa(X)$ be the function field of a variety $X$ over a* perfect *field $k$. Then* $\operatorname{tr.deg.} K/k = \dim X$*, and so $\Omega_{\kappa(X)/k}$ is a vector space of dimension* $\dim X$.

## 5.3 Differential *n*-forms

So far, we have been studying differential 1-forms on a variety. It turns out that the differential $n$-forms are also give very important invariants. In particular, since the space of $n$-forms is one-dimensional, we can associate a divisor to an $n$-form, and all such divisors are linearly equivalent. The divisor class to which they belong is the canonical divisor class.

Let $X$ be a smooth variety over a field, of dimension $n$. By Corollary 5.2.14, the vector space $\Omega_{\kappa(X)/k}$ has dimension $n$. Its $n$th exterior power $\bigwedge^n \Omega_{\kappa(X)/k}$ is therefore a 1-dimensional vector space over $\kappa(X)$; a generator is given by $dx_1 \wedge \cdots \wedge dx_n$, where $x_1, \ldots, x_n$ are any functions such that $dx_1, \ldots, dx_n$ generate $\Omega_{\kappa(X)/k}$.

**Definition 5.3.1** Take an $n$-form $\omega \in \bigwedge^n \Omega_{\kappa(X)/k}$ and a point $P$ on $X$. Let $t_1, \ldots, t_n$ be a set of local parameters at $P$. Then $dt_1, \ldots, dt_n$ generate $\Omega_{\kappa(X)/k}$, and so there is a unique $g \in \kappa(X)$ such that $\omega = g dt_1 \wedge \ldots \wedge dt_n$. We say that $\omega$ is *regular at P* if this $g$ is regular at $P$.

It is straightforward to check that the set of $n$-forms which are regular at $P$ forms a subgroup of $\Omega_{\kappa(X)/k}$. Moreover, if $\omega$ is regular at $P$, and $f \in \kappa(X)$ is a function which is regular at $P$, then $f\omega$ is clearly also regular at $P$.

**Definition 5.3.2** For any open subset $U \subset X$, define an $n$-form $\omega \in \bigwedge^n \Omega_{\kappa(X)/k}$ to be *regular on U* if it is regular at each point of $U$. Let $\omega_X(U)$ denote the set of all such $n$-forms.

By our previous comments, $\omega_X(U)$ is a sub-$\mathscr{O}_X(U)$-module of $\bigwedge^n \Omega_{\kappa(X)/k}$.

*Remark* 5.3.3 Using sheaves, it is more natural to define the sheaf $\omega_X$ as the highest exterior power $\bigwedge^n \Omega_X$ of the sheaf of 1-forms on $X$.

**Definition 5.3.4** The *geometric genus* of a smooth variety $X$ over $k$ is $g(X) = \dim_k \omega_X(X)$.

**Example 5.3.5** For a curve $C$ we have $\Omega_C = \omega_C$, so we have already seen that $\omega_{\mathbf{P}^1}(\mathbf{P}^1) = 0$ and thus $g(\mathbf{P}^1) = 0$.

The fact that $\bigwedge^n \Omega_{\kappa(X)/k}$ is one-dimensional means that $n$-forms on $X$ behave rather like rational functions. Recall that we can associate, to each rational function $f$, a divisor $(f)$. We now show how to do the same for $n$-forms. Firstly, we define the order of vanishing of an $n$-form along a prime divisor.

**Definition 5.3.6** Let $X$ be a smooth variety over a field $k$, let $Z$ be a prime divisor on $X$, and let $P$ be a point of $Z$. Take a non-zero $n$-form $\omega \in \bigwedge^n \Omega_{\kappa(X)/k}$. Let $t_1, \ldots, t_n$ be a set of local parameters at $P$. Then there is a unique $g \in \kappa(X)$ such that $\omega = g dt_1 \wedge \ldots \wedge dt_n$. We define $v_Z(\omega) = v_Z(g)$.

**Exercise 5.3.7** Show that this definition does not depend on the choice of the point $P$.

Now we can put these valuations together to obtain a divisor.

**Definition 5.3.8** Let $X$ be a smooth variety over a field $k$. To any non-zero $n$-form $\omega \in \bigwedge^n \Omega_{\kappa(X)/k}$ we associate the divisor

$$(\omega) = \sum_Z v_Z(\omega) Z \in \operatorname{Div} X,$$

where the summation is over all prime divisors of $X$.

**Exercise 5.3.9** Show that the sum is finite – that is, $v_Z(\omega) = 0$ for all but finitely many prime divisors $Z$.

It is important to realise that the function $g$ in Definition 5.3.6 really does depend on the point $P$, and so there is no reason to expect the divisor $(\omega)$ to be the divisor of any rational function.

For any two non-zero $\omega, \omega' \in \bigwedge^n \Omega_{\kappa(X)/k}$ there is a $g \in \kappa(X)$ such that $\omega = g\omega'$, so $(\omega)$ and $(\omega')$ are linearly equivalent.

**Definition 5.3.10** The class in $\operatorname{Pic} X$ of any, and thus all, $(\omega) \in \bigwedge^n \Omega_{\kappa(X)/k}$ is called the *canonical divisor class* of $X$. The divisors in this class are called *canonical divisors*.

**Exercise 5.3.11** Compute the divisor $(dt)$ on $\mathbf{P}^1(x,y)$ with $t = x/y$.

**Exercise 5.3.12** Compute the divisor $(dx/y)$ on the projective closure in $\mathbf{P}^2$ of the affine curve given by $y^2 = f(x)$ with $f$ a separable polynomial of degree 2, 3, 4, general $d$.

**Exercise 5.3.13** Compute the divisor $(dt_1 \wedge \ldots \wedge dt_n)$ on $\mathbf{P}^n(x_0, x_1, \ldots, x_n)$ with $t_i = x_i/x_0$.

**Exercise 5.3.14** Let $X$ be a hypersurface in $\mathbf{P}^n_k(x_0, \ldots, x_n)$ given by the homogeneous polynomial $F$ of degree $d$, let $L$ be any linear form in $k[x_0, \ldots, x_n]$, and set

$$\omega = \frac{x_0^n L^{-n-1+d}}{\partial F / \partial x_0} dt_1 \wedge \ldots \wedge dt_{n-1}$$

with $t_i = x_i/x_0$. After checking that all degrees work out to make $\omega$ a well-defined element of $\bigwedge^{n-1} \Omega_{\kappa(X)/k}$, show that we have $(\omega) = (-n-1+d)(H \cap X)$, where $H$ is the hyperplane given by $L = 0$.

Note that, with the notation of the previous exercise, there exist $n$-forms that are regular everywhere if and only if $d \geq n+1$, while there are no regular 1-forms if $n > 2$. The following proposition is a generalization of the previous exercise.

**Proposition 5.3.15** *Let $X \subset \mathbf{P}^n$ be a smooth complete intersection of dimension $n-k$, defined by the polynomials $F_1,\dots,F_k$ of degrees $d_1,\dots,d_k$ respectively. Then every canonical divisor on $X$ is linearly equivalent to $(-n-1+\sum_{i=1}^k d_i)H$ where $H$ is any hyperplane section of $X$.*

Proposition 5.3.15 follows from Hartshorne (1977, Proposition II.8.20); for a step-by-step approach, see Exercise II.8.4 there. Besides the sheaf-theoretic proof given there, the following exercises also lead to a (fairly heavily) computational proof.

For any $k$ polynomials $f_1,\dots,f_k \in k[x_1,\dots,x_n]$, and any sequence $J=(j_l)_{l=1}^k$ with $1 \leq j_1 < \dots < j_k \leq n$ we define $M_J = M_J(f_1,\dots,f_k)$ to be the determinant of the matrix $A = (\partial f_i/\partial x_{j_l})_{i,l=1}^k$.

**Exercise 5.3.16** Let $X \subset \mathbf{A}^n(x_1,)$ be a smooth complete intersection of dimension $n-k$, defined by the polynomials $f_1,\dots,f_k \in k[x_1,\dots,x_n]$. Let $J$ be a sequence as above, and let $I$ be the increasing sequence of the elements of $\{1,\dots,n\} \smallsetminus J$. Then up to sign the differential $\omega_J = M_J^{-1} dx_{i_1} \wedge \dots \wedge dx_{i_{n-k}}$ is independent of the choice of $J$.

**Exercise 5.3.17** Use the notation as in the previous exercise, and assume $P$ is a point on $X$. Then there is a particular sequence $J$ as in that exercise such that $M_J(P) \neq 0$ and for the corresponding sequence $I$, the elements $x_i - x_i(P)$ with $i \in I$ form a set of local parameters at $P$. Conclude that $(\omega_J) = 0$ on $X \subset \mathbf{A}^n$.

**Exercise 5.3.18** Homogenize the previous exercises to find out the contribution to $(\omega)$ of the hyperplane at infinity of the projective closure of $X$. Check that your answer agrees with Proposition 5.3.15.

**Exercise 5.3.19** Suppose $X$ is a smooth complete intersection as in Proposition 5.3.15, and assume that $X$ is a surface. Compute the self-intersection of a canonical divisor on $X$.

The following exercise gives another generalization of exercise 5.3.14.

**Exercise 5.3.20** Let $\mathbf{P}(w_0,w_1,\dots,w_n)$ be weighted projective $n$-space with coordinates $x_0,\dots,x_n$ such that $x_i$ has weight $w_i$, and assume $w_0 = 1$. Let $X$ be a smooth hypersurface in $\mathbf{P}(w_0,w_1,\dots,w_n)$ of (weighted) degree $d$. Set $D = X \cap H$ where $H$ is the hyperplane given by $x_0 = 0$. Then any canonical divisor on $X$ is linearly equivalent to $(d - \sum_i w_i)D$.

**Exercise 5.3.21** Find an example of a variety $X$ of dimension $n$ for which the map $\bigwedge^n(\Gamma(\Omega_X)) \to \Gamma(\omega_X)$ is not surjective.

# 6

# Linear systems and the Riemann–Roch Theorem

## 6.1 Equivalent effective divisors

Let $X$ be a smooth variety over a field $k$, and let $D$ be any divisor on $X$. We define the $k$-vector space

$$L(D) = \{f \in \kappa(X)^\times \mid (f) + D \text{ is effective }\} \cup \{0\}. \tag{6.1}$$

Given a nonzero function $f \in L(D)$, the definition says that $D' = (f) + D$ is effective; but it is also linearly equivalent to $D$. Conversely, suppose that $D'$ is an effective divisor on $X$ linearly equivalent to $D$; then there is a function $f \in \kappa(X)^\times$ such that $D' = D + (f)$, and so $f \in L(D)$. So there is a correspondence

$$L(D) \smallsetminus \{0\} \longrightarrow \{\text{Effective divisors linearly equivalent to } D\}.$$

If $X$ is projective, then Proposition 4.1.8 says that two functions $f, f'$ give rise to the same divisor $(f) = (f')$ if and only if their quotient is constant. In that case there is a *bijective* correspondence

$$\frac{L(D) \smallsetminus \{0\}}{k^\times} \longleftrightarrow \{\text{Effective divisors linearly equivalent to } D\}. \tag{6.2}$$

If $L(D)$ is a finite-dimensional vector space, then the left-hand side of this correspondence can be thought of as the projective space $\mathbf{P}(L(D))$.

**Definition 6.1.1** The set of divisors in (6.2) is called the *complete linear system* associated to the divisor class $[D]$.

**Example 6.1.2** Let $X$ be $\mathbf{P}^2_k$ with coordinates $X, Y, Z$ and take $D$ to be the line $\{X = 0\}$. Then

$$L(D) = \left\{ \frac{aX + bY + cZ}{X} \mid a, b, c \in k \right\}.$$

Here $L(D)$ has dimension 3, and the effective divisors linearly equivalent to $D$

are all divisors defined by $aX+bY+cZ=0$ for $a,b,c\in k$ – that is, they are the straight lines in $\mathbf{P}^2$. They are parametrised by a projective space of dimension 2, called the *dual* of $\mathbf{P}^2$.

In this example, the space $L(D)$ is finite-dimensional. However, this does not have to be true if $X$ is not projective.

**Example 6.1.3** Let $X$ be $\mathbf{A}^1_k$ with coordinate $t$, and let $D$ be the divisor $\{t=0\}$. Then

$$L(D)=\{t^{-1}P(t)\mid P\in k[t]\}$$

which has infinite dimension. Intuitively, this can happen because $\mathbf{A}^1$ is "missing" the point at infinity, so the functions in $L(D)$ may have poles of arbitrary degree there.

**Proposition 6.1.4** *Let $X$ be a projective variety, and let $D$ be a divisor on $X$. Then $L(D)$ is finite-dimensional.*

*Proof* See Hartshorne (1977, Theorem II.5.19). □

**Definition 6.1.5** Let $X$ be a projective variety over a field $k$ and let $D$ be a divisor on $X$. Define $\ell(D):=\dim_k L(D)$ to be the dimension of the vector space (6.1).

**Proposition 6.1.6** *Let $X$ be a smooth projective surface over a field $k$, let $D$ be a divisor on $X$ and let $H$ be a hyperplane section on $X$. Suppose that $D\cdot H<0$. Then $\ell(D)=0$.*

*Proof* If $\ell(D)\geq 1$, then there would be an effective divisor $D'$ linearly equivalent to $D$, such that $D'\cdot H<0$. Since any prime divisor $Z$ satisfies $Z\cdot H=\deg Z>0$, this is a contradiction. □

**Example 6.1.7** Let $C$ be a curve lying in a smooth projective surface $X$. If $C$ is linearly equivalent to another curve $C'$ which meets $C$ transversely, then $C^2=C\cdot C'\geq 0$. But a curve does not have to have positive self-intersection. For example, let $X$ be a smooth cubic surface in $\mathbf{P}^3_{\mathbf{C}}$ and let $C$ be one of the 27 straight lines lying on $X$. If there were any other effective divisor on $X$ linearly equivalent to $C$, then $\ell(C)>1$ and there would be a whole infinite family of effective divisors $C'$ linearly equivalent to $C$. Since they would all satisfy $C'\cdot H=1$, they would all be straight lines – yet there are only finitely many straight lines in $X$, so this cannot be true. In fact, as we shall see, $C^2=-1$ for a straight line $C$ lying in a smooth cubic surface.

**Exercise 6.1.8** Show that, if $D'\sim D$, then the vector spaces $L(D')$ and $L(D)$ are isomorphic.

## 6.2 Rational maps to projective space

In this section, we show how divisor classes can give rise to rational maps into projective space. For more details about the subject of this section, see Hartshorne (1977, Section II.7).

Fix a divisor $D$ on a variety $X$, and choose a basis $(f_1, \ldots, f_d)$ of $L(D)$, where $d = \ell(D)$. Then these functions may be used to define a rational map

$$\phi_D \colon X \to \mathbf{P}^{d-1}, \quad P \mapsto [f_1(P) : \cdots : f_d(P)].$$

This map is regular wherever the $f_i$ are all defined and do not all vanish. It may be possible to clear denominators and so extend the domain of $\phi_D$.

*Remark* 6.2.1 We have defined $\phi_D$ by choosing a basis for $L(D)$. Choosing a different basis gives a map which differs from $\phi_D$ by a linear automorphism of $\mathbf{P}^{d-1}$. So, in fact, $\phi_D$ is defined only up to automorphisms of $\mathbf{P}^{d-1}$. It is, however, possible to avoid this ambiguity by taking the domain of $\phi_D$ to be $\mathbf{P}(L(D))$, the projective space of dimension $\ell(D) - 1$ associated to the vector space $L(D)$.

Suppose that $D'$ is linearly equivalent to $D$, say $D' = D + (f)$. Then, given any basis $\{f_i\}$ for $L(D)$, we obtain a basis $\{f^{-1} f_i\}$ for $L(D')$. These bases define *the same* rational map, and so the map $\phi_D$ really depends only on the linear equivalence class of $D$. We will often speak of the rational map associated to a divisor *class*, rather than to an individual divisor.

**Example 6.2.2** Let $X$ be $\mathbf{P}^1_k$, with coordinates $s,t$, and let $P$ be the point $s = 0$. Then a basis for $L(P)$ is given by $\{1, t/s\}$, and we obtain the rational map $\phi_P = [1 : t/s] = [s : t]$, the identity morphism.

Now consider the divisor $2P$. A basis for $L(2P)$ is given by $\{1, t/s, t^2/s^2\}$, and the corresponding rational map is $\phi_{2P} = [s^2 : st : t^2]$, which embeds $\mathbf{P}^1$ as the conic curve $\{y^2 = xz\}$ in $\mathbf{P}^2$. Observe that any pair of points makes an effective divisor equivalent to $2P$, and these are precisely the divisors cut out by hyperplane sections under this embedding.

Similarly, we obtain $\phi_{3P} = [s^3 : s^2 t : st^2 : t^3]$, embedding $\mathbf{P}^1$ as the *twisted cubic curve* in $\mathbf{P}^3$.

**Example 6.2.3** Let $X$ be a projective subvariety of $\mathbf{P}^n$, with coordinates $X_0, \ldots, X_n$. Let $H \subset \mathbf{P}^n$ be the hyperplane given by $X_0 = 0$ and set $D = X \cap H$. The functions $X_i/X_0$ are contained in $L(D)$, and the rational map corresponding to $D$ is the given embedding of $X$ in $\mathbf{P}^n$. Tautologically, the hyperplane sections under this embedding are the effective divisors linearly equivalent to $D$.

The functions $X_iX_j/X_0^2$ form a basis for $L(2D)$ and determine the *2-uple embedding* of $X$ in $\mathbf{P}^N$, with $N = \binom{n+2}{2} - 1$.

*Remark* 6.2.4 If the functions $f_i$ do not span the whole of $L(D)$, we still obtain a rational map. For example, take $X = \mathbf{P}^1$ as in example 6.2.2 and $D = 3P$, but take the functions $\{1, t^2/s^2, t^3/s^3\}$. The corresponding map $[s^3 : st^2 : t^3]$ maps $\mathbf{P}^1$ to the cuspidal cubic curve $\{y^3 = xz^2\}$. But there is really nothing new here – this map can equally be obtained by embedding $\mathbf{P}^1$ as the twisted cubic in $\mathbf{P}^3$, and then projecting to $\mathbf{P}^2$ in an appropriate way.

**Exercise 6.2.5** Let $C$ be the curve in $\mathbf{P}^3(x,y,z,w)$ parametrized by $(u^4 : u^3t : ut^3 : t^4)$. Let $H$ be the hyperplane given by $w = 0$ and set $D = C \cap H$. Show that the functions $1, x/w, y/w, z/w$ do not generate $L(D)$. Find a divisor which is linearly equivalent to $D$, but is not a hyperplane section. (Hint: find an isomorphism from $C$ to $\mathbf{P}^1$ and find what divisor $D$ corresponds to on $\mathbf{P}^1$.)

The next proposition will be needed in an exercise. Note that any smooth variety is normal, and that a variety defined by a single equation (a hypersurface) is a complete intersection.

**Proposition 6.2.6** *Let $X$ be a normal complete intersection in $\mathbf{P}^n$ and $H$ a hyperplane in $\mathbf{P}^n$. Then the map $L(nH) \to L(nH \cap X)$ is surjective.*

*Proof* See Hartshorne (1977, Exercise II.8.4(c)). □

**Exercise 6.2.7** Use Proposition 6.2.6 to show that the geometric genus of a hypersurface in $\mathbf{P}^n$ of degree $d$ equals $\binom{d-1}{n}$.

**Example 6.2.8** Let $\omega_0, \ldots, \omega_r$ be a basis for the vector space of regular differentials on $X$. Since $\bigwedge^n \Omega_{k(X)/k}$, with $n = \dim X$, is 1-dimensional over $k(X)$, there are rational functions $g_1, \ldots, g_r$ such that $\omega_i = g_i\omega_0$. We can define an associated rational map by $P \mapsto [1 : g_1(P) : \cdots : g_r(P)]$.

**Exercise 6.2.9** Show that the $g_i$ in Example 6.2.8 are a basis for $L(D)$ with $D = (\omega_0)$.

## 6.3 Ample and very ample divisors

The hyperplane sections of a projective variety are very useful in studying its geometry. However, we have seen that a variety may admit many maps to projective spaces, and even many embeddings into projective spaces; so the idea of “hyperplane section” is not intrinsically defined. The notions of ample and very ample divisors are ones which embody some of the properties of hyperplane sections.

**Definition 6.3.1** A divisor $D$ on a projective variety $X$ is *very ample* if a basis of $L(D)$ determines a morphism $X \to \mathbf{P}^n$ that is an embedding.

**Exercise 6.3.2** Show that any divisor that is linearly equivalent to a very ample divisor, is in fact itself very ample.

**Example 6.3.3** If $X$ is embedded in $\mathbf{P}^n$, then any hyperplane section of $X$ is a very ample divisor.

Conversely, every very ample divisor on $X$ is of this form for some embedding $X \to \mathbf{P}^r$.

**Definition 6.3.4** A divisor $D$ on $X$ is called ample is some positive multiple of $D$ is very ample.

**Example 6.3.5** Let $C$ be a smooth curve. Then any divisor of positive degree is ample, by the Riemann–Roch Theorem for curves . However, a divisor of degree 1 is very ample if and only if $C$ has genus 0.

**Example 6.3.6** Consider the cone $X$ given by $x^2 + y^2 = z^2$ in $\mathbf{P}^3$. Show that any two lines on $X$ through the vertex of $X$ are linearly equivalent. Show that each of these lines is ample, but not very ample. (Ok, this is cheating, as we said $X$ would always be smooth; later we will see del Pezzo surfaces, for some of which the anticanonical sheaf is ample, yet not very ample).

**Exercise 6.3.7** Find all sequences $(d_1, \ldots, d_r)$ with $d_i \geq 2$ such that a canonical divisor on a smooth complete intersection $X$ in $\mathbf{P}^{r+2}$ of hypersurfaces of degree $d_1, \ldots, d_r$ is not very ample. (Compare this to the next lecture.)

## 6.4 Arithmetic genus and the adjunction formula

Any projective variety $X$ has an integer invariant, written $p_a(X)$, called the *arithmetic genus* of $X$. This may be defined in several equivalent ways: for example, in terms of the Euler characteristic of the structure sheaf of $X$ by

$$p_a(X) = (-1)^{\dim X}(\chi(\mathscr{O}_X) - 1)$$

or using the Hilbert polynomial of $X$. See Hartshorne (1977, Chapter I, Exercise 7.2).

**Proposition 6.4.1** *Let $C$ be a smooth, projective curve. Then $p_a(C) = p_g(C)$, the geometric genus of $C$.*

*Proof* See Hartshorne (1977, Chapter IV, Proposition 1.1). □

*Remark* 6.4.2 If $k = \mathbf{C}$, then $C(\mathbf{C})$ is a Riemann surface with a well-defined topological genus. The topological genus is equal to the geometric genus.

**Theorem 6.4.3** (Adjunction formula) *Let $C$ be a (possibly singular) curve in a smooth projective surface $X$. Let $K$ be a canonical divisor on $X$. Then*

$$2p_a(C) - 2 = C \cdot (C + K).$$

*Proof* For non-singular curves, this is Hartshorne (1977, Chapter V, Proposition 1.5). The general case will follow easily from the Riemann–Roch Theorem. □

*Remark* 6.4.4 In particular, the adjunction formula shows that linearly equivalent effective divisors have the same arithmetic genus. Thus all curves of degree 3 in $\mathbf{P}^2$ have arithmetic genus 1, whether they are smooth or not. For example, the smooth cubic curve $y^2 = x(x-1)(x-2)$ has arithmetic genus 1, and indeed geometric genus 1. The nodal cubic $y^2 = x(x-1)^2$ and the cuspidal cubic $y^2 = x^3$ both also have arithmetic genus 1, although they are birational to $\mathbf{P}^1$ which has arithmetic and geometric genus 0.

*Remark* 6.4.5 The previous remark can be viewed as a special case of the following fact: arithmetic genus is constant in "continuously varying families" of projective varieties, by which we mean that it is locally constant on the fibres of a flat proper morphism. See Hartshorne (1977, Chapter III, Corollary 9.10) for further reading.

**Exercise 6.4.6** Let $C$ be a curve of degree $d$ in $\mathbf{P}^2$. Show that $p_a(C) = (d-1)(d-2)/2$.

## 6.5 The Riemann–Roch Theorem

In many situations, it is very useful to know the number $\ell(D)$ associated to a divisor $D$, but so far we have no way of computing it in any but the simplest examples. The Riemann–Roch Theorem relates $\ell(D)$ to some intersection numbers. We will need one more definition.

*Remark* 6.5.1 There is another vector space which can be associated to a divisor $D$, the first sheaf cohomology group of $X$ with values in $\mathscr{O}(D)$. We will rarely need to know anything about this vector space, other than that its dimension is (of course) non-negative. The reason we mention it at all is that it makes an appearance in the statement of the Riemann–Roch Theorem.

**Definition 6.5.2** Let $X$ be a smooth projective surface, and $D$ a divisor on $X$. The *superabundance* of $D$, written $s(D)$, is defined by $s(D) = \dim \mathrm{H}^1(X, \mathscr{O}(D))$.

**Theorem 6.5.3** (Riemann–Roch) *Let $X$ be a smooth, geometrically irreducible, projective surface over a field $k$. Let $K$ be a canonical divisor on $X$, and let $D$ be any divisor on $X$. Then*

$$\ell(D) - s(D) + \ell(K - D) = \tfrac{1}{2} D \cdot (D - K) + 1 + p_a(X).$$

*Proof* See Hartshorne (1977, Chapter V, Theorem 1.6). Although there the theorem is proved under the assumption that $k$ is algebraically closed, all the quantities involved are preserved by moving from $k$ to its algebraic closure: see Kollár et al. (2004, Exercise 3.34). □

**Example 6.5.4** Let $X$ be a smooth, geometrically irreducible, projective surface, let $H$ be a hyperplane section of $X$, and let $D$ be any divisor on $X$. Then, for $n$ sufficiently large, $D + nH$ is equivalent to an effective divisor.

*Proof* Recall that $H^2 = \deg X > 0$ from Example 4.3.8. We deduce that

$$(K - (D + nH)) \cdot H = K \cdot H - D \cdot H - n \deg X$$

is negative for $n$ sufficiently large. Under that condition, applying the Riemann–Roch Theorem to $D + nH$ gives

$$\begin{aligned} \ell(D + nH) &\geq \tfrac{1}{2}(D + nH) \cdot (D + nH - K) + 1 + p_a(X) \\ &\geq \tfrac{1}{2}(n^2 \deg D + n(2 \deg D - H \cdot K) + D^2 - D \cdot K) + 1 + p_a(X) \end{aligned}$$

which is at least 1 for $n$ sufficiently large, and therefore $D + nH$ is linearly equivalent to an effective divisor. □

*Remark* 6.5.5 This can easily be generalised by replacing $H$ with any ample divisor.

**Exercise 6.5.6** Let $X$ be a smooth, rational surface which can be embedded in $\mathbf{P}^n$ by its anticanonical divisor, i.e. such that $K_X = -H$ where $H$ is a hyperplane section. Such a surface is a del Pezzo surface. In this case, $p_a(X) = 0$. For example, $X$ could be a cubic surface in $\mathbf{P}^3$ or the intersection of two quadrics in $\mathbf{P}^4$.

(i) Let $C$ be a smooth curve in $X$. Use the adjunction formula to show that $C^2 \geq -1$.
(ii) Conversely, let $D$ be a divisor satisfying $D^2 \geq -1$ and $D.H > 0$. Use the Riemann–Roch Theorem to show that the linear equivalence class of $D$ contains an effective divisor.

(iii) Let $D$ be a divisor on $X$ such that $nD$ is principal for some positive integer $n$. Show that $D^2 = 0$. Using the Riemann–Roch theorem, deduce that $D \sim 0$ and therefore that $\operatorname{Pic} X$ is torsion-free.

# 10

# The Brauer group of a field

In this chapter we introduce the Brauer group of a field, defined to be the group of equivalence classes of central simple algebras over the field.

There are several clear and thorough references on Brauer groups: among them are Gille and Szamuely (2006); Bourbaki (2012); and Chapter IV of Milne (2008).

## 10.1 Hilbert symbols

Before defining the Brauer group of a general field, we introduce the Hilbert symbol. The Hilbert symbol is closely related to the Legendre symbol for quadratic residues, and it allows us to reformulate the counterexamples to the Hasse principle seen in Chapter 2 in a more uniform way. As we shall see later, this can be viewed as an explicit way of performing computations with quaternion algebras over number fields. Serre (1973, Chapter III) is an excellent reference when the field is the rational numbers, and everything can be made very explicit; see Milne (2008, Section III.4) for a thorough treatment of the general case.

**Definition 10.1.1** Let $k$ be a number field, let $v$ be a place of $k$, and let $a$ and $b$ be two elements of $k_v^\times$. We define the *Hilbert symbol* $(a,b)_v$ of $a$ and $b$ at $v$ to be

$$(a,b)_v = \begin{cases} 1 & \text{if the conic } ax^2+by^2=z^2 \text{ has a } k_v\text{-rational point;} \\ -1 & \text{otherwise.} \end{cases}$$

While this definition might appear rather *ad hoc*, the following results show that Hilbert symbols have many pleasant algebraic properties.

**Proposition 10.1.2** *Let $k$ be a number field and let $v$ be a place of $k$. The Hilbert symbol has the following properties.*

(i) *Symmetry: for all $a,b \in k_v^\times$, $(a,b)_v = (b,a)_v$.*
(ii) *If either $a$ or $b$ is a square in $k_v$, then $(a,b)_v$ equals 1.*
(iii) *For all $a,b,c \in k_v^\times$, we have $(a,b)_v = (ac^2,b)_v$. That is, $(a,b)_v$ only depends on the class of $a$ in $k_v^\times/(k_v^\times)^2$.*
(iv) *For all $a \in k_v^\times$, $(a,-a)_v = 1$.*
(v) *If $a \neq 0,1$, then $(a,1-a)_v = 1$.*

*Proof* These are all immediate from the definition of the Hilbert symbol. □

**Proposition 10.1.3** *Let $k$ be a number field, let $v$ be a place of $k$ and let $a,b \in k_v^\times$. The equality $(a,b)_v = 1$ holds if and only if $a$ is a norm from $k_v(\sqrt{b})$.*

*Remark* 10.1.4 In some ways it would be more natural to use the algebra $k_v[t]/(t^2-b)$ instead of $k_v(\sqrt{b}$, whether or not $b$ is a square. The distinction doesn't matter, though, since an element $a \in k_v$ is a norm from $k_v(\sqrt{b})$ if and only if it is a norm from $k_v[t]/(t^2-b)$. The only case where this needs to be checked is when $b$ is a square, in which case both norm maps are easily seen to be surjective.

*Proof of Proposition 10.1.3* In view of the above remark, we must prove: $(a,b)_v = 1$ is and only if $a$ is a norm from $k_v[t]/(t^2-b)$. Suppose first that $a$ is the norm of $\alpha+\beta t \in k_v[t]/(t^2-b)$. Then we have $a = \alpha^2 - b\beta^2$, and so $(x,y,z) = (1,\beta,\alpha)$ is a solution to $ax^2+by^2 = z^2$, showing that $(a,b)_v$ is 1.

Conversely, suppose that $x,y,z \in k_v$ satisfy $ax^2+by^2 = z^2$, with $x,y,z$ not all zero. If $x$ is non-zero, then we have $a = (z/x)^2 - b(y/x)^2$ and so $a$ is the norm of $(z/x)+(y/x)t$. If $x$ is zero, then $y$ is non-zero and $b = (z/y)^2$ is a square, in which case the norm map $k_v[t]/(t^2-b) \to k_v$ is surjective. □

As a corollary, we can prove the easy part of the bilinearity of $(a,b)_v$.

**Corollary 10.1.5** *Suppose that $a,b \in k_v^\times$ satisfy $(a,b)_v = 1$. Then, for all $c \in k_v^\times$, we have $(c,b)_v = (ac,b)_v$.*

*Proof* If $a$ is a norm from $k_v(\sqrt{b})$, then $c$ is a norm if and only if $ac$ is a norm. □

There are simple explicit formulae to evaluate the Hilbert symbol for any completion of **Q**.

**Proposition 10.1.6** (i) *Let $a, b \in \mathbf{R}^\times$. Then*

$$(a,b)_\infty = \begin{cases} 1 & \text{if } a > 0 \text{ or } b > 0; \\ -1 & \text{if } a < 0 \text{ and } b < 0. \end{cases}$$

(ii) *Let $p$ be an odd prime; let $a, b \in \mathbf{Q}_p^\times$, and write $a = p^\alpha u$ and $b = p^\beta v$ with $u, v \in \mathbf{Z}_p^\times$. Write $\varepsilon(p) = (p-1)/2$. Then*

$$(a,b)_p = (-1)^{\alpha\beta\varepsilon(p)} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha.$$

*In particular, $(u,v)_p = 1$ if $u, v \in \mathbf{Z}_p^\times$.*

(iii) *Let $a, b \in \mathbf{Q}_2^\times$ and write $a = 2^\alpha u$ and $b = 2^\beta v$ with $u, v \in \mathbf{Z}_2^\times$. For $x \in \mathbf{Z}_2^\times$, write $\varepsilon(x) = (x-1)/2 \pmod 2$ and $\omega(x) = (x^2-1)/8 \pmod 2$. Then*

$$(a,b)_2 = (-1)^{\varepsilon(u)\varepsilon(v) + \alpha\omega(v) + \beta\omega(u)}.$$

*Proof* See Serre (1973, Chapter III, Theorem 1). □

We now describe some deeper properties of Hilbert symbols.

**Proposition 10.1.7** *Let $k$ be a number field.*

(i) *For each place $v$ of $k$, the Hilbert symbol defines a non-degenerate symmetric bilinear form on the $\mathbf{F}_2$-vector space $k_v^\times/(k_v^\times)^2$.*

(ii) *For $a, b \in k^\times$, we have the* product formula

$$\prod_v (a,b)_v = 1$$

*where the product is taken over all places of $k$.*

*Proof* For direct proofs of these statements in the case $k = \mathbf{Q}$, using the description of Proposition 10.1.6, see Serre (1973, Chapter III, Theorems 2 and 3). In the general case, we outline how to deduce the statements from class field theory.

Statement (i), while it looks simple, rests on two highly non-trivial results of local class field theory for quadratic extensions. Given Corollary 10.1.5, to prove bilinearity of the Hilbert symbol it is enough to show $(ac, b) = 1$ whenever we have $(a,b) = -1$ and $(c,b) = -1$. By Proposition 10.1.3 it suffices to show that, whenever $b \in k_v^\times$ is non-square, the product of two non-norms from $k_v(\sqrt{b})$ is a norm. This follows from the existence of the local reciprocity map (Milne, 2008, Theorem I.1.1), which gives an isomorphism $k_v^\times / \mathrm{N}_{k_v(\sqrt{b})/k_v}(k_v(\sqrt{b})^\times) \to \mathrm{Gal}(k_v(\sqrt{b})/k_v)$.

To prove that the bilinear form defined by the Hilbert symbol is non-degenerate we must show that, if an element $a \in k_v^\times$ is not a square, then there exists $b \in k_v^\times$

satisfying $(a,b)_v = -1$. This follows from the Existence Theorem in local class field theory (Milne, 2008, Theorem I.1.4). Suppose that $a$ is a non-square in $k_v^\times$. Let $U$ be a complement in the $\mathbf{F}_2$-vector space $k_v^\times/(k_v^\times)^2$ to the subspace $\{1,a\}$ and let $\tilde{U}$ be the inverse image of $U$ in $k^\times$. Then $\tilde{U} \subset k_v^\times$ is a subgroup of index 2 in $k_v^\times$ not containing $a$. By the Existence Theorem, there exists a quadratic extension $\ell/k_v$ satisfying $\tilde{U} = \mathrm{N}_{\ell/k_v}(\ell^\times)$, and so $a$ is not a norm from $\ell$. We have $\ell = k_v(\sqrt{b})$ for some non-square $b \in k_v^\times$, and by Proposition 10.1.3 it follows that $(a,b)_v$ is $-1$.

Statement (ii), the product formula for the Hilbert symbol, is a consequence of the global reciprocity law of class field theory (Milne, 2008, Theorem V.5.3). Let $a$ and $b$ be elements of $k^\times$. If $b$ is a square, then the formula is trivial; so assume that $b$ is non-square. Let $\ell$ be the field $k(\sqrt{b})$. For each place $v$ of $k$, the local symbol $(a,b)_v$ is identified (in the obvious way) with the image of $a$ under the local reciprocity map $\theta_v \colon k_v^\times/\mathrm{N}_{\ell_w/k_v}(\ell_w^\times) \to \mathrm{Gal}(\ell_w/k_v)$, where $w$ is a place of $\ell$ lying over $v$. Let $\mathbf{I}_k$ denote the group of idèles of $k$. The global reciprocity map $\theta \colon \mathbf{I}_k/\mathrm{N}_{\ell/k}\mathbf{I}_\ell \to \mathrm{Gal}(\ell/k)$ is the product of the local reciprocity maps and is trivial on $k^\times \subset \mathbf{I}_k$, giving $\prod_v (a,b)_v = 1$. □

The following long exercise shows how to formulate Example 2.3.5 in terms of norms and Hilbert symbols.

**Exercise 10.1.8** Let $X$ be a smooth, projective, geometrically irreducible variety over $\mathbf{Q}$. Let $f \in \kappa(X)^\times$ be a rational function on $X$, and suppose that there exists a quadratic extension $k = \mathbf{Q}(\sqrt{d})$ of $\mathbf{Q}$ such that the divisor $(f)$ is a norm from $k$ – that is, there is some divisor $D \in \mathrm{Div}\, X_k$ satisfying $(f) = \mathrm{N}_{k/\mathbf{Q}} D = D + \sigma(D)$, where $\sigma$ is the generator of $\mathrm{Gal}(k/\mathbf{Q})$.

Let $K$ be a field containing $\mathbf{Q}$; we aim to construct a function

$$\phi_K \colon X(K) \to K^\times/\mathrm{N}_{K(\sqrt{d})/K}(K(\sqrt{d})^\times)$$

associated to $f$.

Because $X$ is geometrically irreducible, the base field $k$ is algebraically closed in the function field $\kappa(X)$ **??**, and so the extension $\kappa(X_k)/\kappa(X)$ is a quadratic extension generated by a square root of $d$. To save on notation, we will write $\mathrm{N}_{k/\mathbf{Q}}$ not only for the norm map from $k$ to $\mathbf{Q}$, but also for the norm map from $\kappa(X_k)$ to $\kappa(X)$.

(i) On the subset of $X(K)$ consisting of points at which $f$ is regular and non-zero, define a function $\phi_{K,f}$ by setting $\phi_{K,f}(x)$ to be the class of $f(x)$ in $K^\times/\mathrm{N}_{K(\sqrt{d})/K}(K(\sqrt{d})^\times)$. Show that multiplying $f$ by a norm does not affect the function $\phi_{K,f}$. In other words, if $g \in \kappa(X_k)^\times$ is a rational function on $X_k$, and $f'$ is defined to be $\mathrm{N}_{k/\mathbf{Q}}(g)f$, show that the functions $\phi_{K,f}$ and $\phi_{K,f'}$ agree at the points where they are both defined.

(ii) Let $P$ be a point in $X(K)$. Show that there exists a function $g \in \kappa(X_k)^\times$ such that $f' = \mathrm{N}_{k/\mathbf{Q}}(g)f$ has neither a zero nor a pole at $P$. Deduce that various functions $\phi_{K,f'}$ patch together to produce a well-defined function $\phi_K$ as desired.

(iii) Taking $K = \mathbf{Q}_v$, we get a map which we can equally well write using the Hilbert symbol:

$$X(\mathbf{Q}_v) \to \{\pm 1\}, \qquad x \mapsto (d, \phi_{\mathbf{Q}_v}(x))_v.$$

Multiplying these maps, we get a map from the product of all the $X(\mathbf{Q}_v)$ to $\{\pm 1\}$ as follows:

$$\prod_v X(\mathbf{Q}_v) \to \{\pm 1\}, \qquad (x_v) \mapsto \prod_v (d, \phi_{\mathbf{Q}_v}(x_v))_v.$$

Show that the diagonal image of $X(\mathbf{Q})$ in $\prod_v X(\mathbf{Q}_v)$ must lie in the kernel of this map (i.e. the inverse image of 1).

(iv) Consider the surface described by Birch and Swinnerton-Dyer given in Example 2.3.5. Show the the divisor of the function $f = u/(u+v)$ is a norm from $\mathbf{Q}(\sqrt{5})$.

(v) Calculate the maps of (iii), and deduce that $X(\mathbf{Q})$ is empty.

## 10.2 Central simple algebras

The Brauer group of a field is an invariant that plays an important rôle in class field theory. It is also essential for defining the Brauer–Manin obstruction. The Brauer group consists of equivalence classes of central simple algebras, and we begin by studying these algebras.

An *algebra* over a field $k$ is a vector space $A$ over $k$ endowed with a $k$-bilinear multiplication $A \times A \to A$ making $A$ into a ring with identity $1_A$. If $A$ is non-zero, then the map $k \to A$ sending $x$ to $x.1_A$ is an injective ring homomorphism with image contained in the centre of $A$. Note that we require the multiplication on $A$ to be associative, but not necessarily commutative. If $\{a_\alpha\}_{\alpha \in I}$ is a basis of $A$, then to define the multiplication on $A$ it is sufficient to define multiplication on the $a_\alpha$, by specifying the *structure constants* $\{c^\gamma_{\alpha\beta}\}_{\alpha,\beta,\gamma \in I}$ satisfying $a_\alpha a_\beta = \sum_\gamma c^\gamma_{\alpha\beta} a_\gamma$. An algebra is a *division algebra* if every non-zero element has a multiplicative inverse.

**Definition 10.2.1** Let $k$ be a field and $A$ an algebra over $k$. The algebra $A$ is *central* if the centre of $A$ is $k$. The algebra $A$ is *simple* if it has exactly two

two-sided ideals, namely $\{0\}$ and $A$ itself. A *central simple algebra* over $k$ is a finite-dimensional $k$-algebra that is both central and simple.

Note that a central simple algebra is, by definition, finite-dimensional in addition to being central and simple. Easy examples of central simple algebras are matrix algebras: for any field $k$ and any natural number $n$, the $k$-algebra $\mathrm{M}_n(k)$ of $n \times n$ matrices with coefficients in $k$ is a central simple algebra. More generally for any ring $R$, let $\mathrm{M}_n(R)$ denote the ring of $n \times n$ matrices over $R$.

**Exercise 10.2.2** Let $R$ be a ring, and let $m, n$ be positive integers.

(i) Show that the $R$-algebras $\mathrm{M}_m(\mathrm{M}_n(R))$ and $\mathrm{M}_{mn}(R)$ are isomorphic.
(ii) Show that, if $S$ is an $R$-algebra, then there is a natural isomorphism $\mathrm{M}_n(R) \otimes_R S \to \mathrm{M}_n(S)$.
(iii) If $I$ is an ideal of $R$, let $\mathrm{M}_n(I)$ denote the subset of $\mathrm{M}_n(R)$ consisting of matrices with entries in $I$. Show that $I \leftrightarrow \mathrm{M}_n(I)$ is a bijection between the set of two-sided ideals of the ring $R$ and the set of two-sided ideals of $\mathrm{M}_n(R)$.
(iv) Let Id denote the identity matrix in $\mathrm{M}_n(R)$. Show that the map $R \to \mathrm{M}_n(R)$ defined by $r \mapsto r\,\mathrm{Id}$ identifies the centre of $R$ with the centre of $\mathrm{M}_n(R)$.
(v) Conclude that, if $A$ is a central simple algebra over a field $k$, then $\mathrm{M}_n(A)$ is also a central simple algebra over $k$.

It is easy to see that every division algebra is simple, and so every finite-dimensional central division algebra over $k$ is a central simple algebra over $k$. The matrix algebra $\mathrm{M}_n(k)$ is not a division algebra for $n \geq 2$, since it contains non-invertible matrices. A central simple algebra over $k$ is called *split* if it is isomorphic to $\mathrm{M}_n(k)$ for some $n > 0$.

To see an example of a non-split central division algebra, take $k = \mathbf{R}$, and let $\mathbb{H}_{\mathbf{R}}$ be the $\mathbf{R}$-algebra of Hamilton quaternions: the underlying vector space of $\mathbb{H}_{\mathbf{R}}$ is $\mathbf{R}^4$ with basis $(1, i, j, ij)$, and multiplication is uniquely determined by

$$i^2 = -1, \quad j^2 = -1, \quad ji = -ij.$$

Note that the relation $(ij)^2 = -1$ is a consequence of the above.

**Exercise 10.2.3** Show that $\mathbb{H}_{\mathbf{R}}$ is a division algebra over $\mathbf{R}$.

We did not use many properties of the field $\mathbf{R}$ in the preceding example: the fact that every element of $\mathbb{H}_{\mathbf{R}}$ has an inverse follows essentially from the fact that $-1$ is not a square in $\mathbf{R}$. This leads us to the more general definition of a quaternion algebra.

**Definition 10.2.4** Let $k$ be a field of characteristic different from 2 and let $a,b$ be non-zero elements of $k$. Define the *quaternion algebra* $(a,b)_k$ to be the $k$-algebra whose underlying vector space is $k^4$ with basis $(1,i,j,ij)$ and on which multiplication is uniquely determined by

$$i^2 = a, \quad j^2 = b, \quad ji = -ij.$$

As in the case of the algebra $\mathbb{H}_{\mathbf{R}}$, the relation $(ij)^2 = -ab$ follows from the definitions.

The notation for a quaternion algebra is very similar to that for a Hilbert symbol; we shall see (Exercise 10.4.4) that the two notions are closely related.

**Exercise 10.2.5** Let $k$ be a field of odd characteristic, and let $a,b,c$ be elements of $k^\times$.

(i) Prove that the quaternion algebra $(a,b)_k$ is a central simple algebra over $k$.
(ii) Prove that the three quaternion algebras $(a,b)_k$, $(b,a)_k$ and $(ac^2,b)_k$ are isomorphic.
(iii) Prove that the quaternion algebra $(a^2,b)_k$ is not a division algebra.
(iv) Suppose that $a$ is not a square in $k$ and let $\gamma \in k(\sqrt{a})^\times$. Prove that the algebras $(a,b)_k$ and $\big(a, b\mathrm{N}_{k(\sqrt{a})/k}(\gamma)\big)$ are isomorphic.

**Example 10.2.6** Let $A$ be the quaternion algebra $(-1,3)_{\mathbf{F}_7}$. We will show that this algebra is split by determining an explicit isomorphism $\rho: A \to \mathrm{M}_2(\mathbf{F}_7)$. It suffices to find two matrices $I,J \in \mathrm{M}_2(\mathbf{F}_7)$ satisfying the equations $I^2 = -\mathrm{Id}$, $J^2 = 3\mathrm{Id}$ and $IJ + JI = 0$, since the assignment $\rho(i) = I$ and $\rho(j) = J$ will then define a homomorphism of algebras; because $A$ is simple, $\rho$ is then injective and hence surjective by considering dimensions. The matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

satisfies $I^2 = -\mathrm{Id}$; we try to find a compatible $J$ of the form

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The condition $IJ + JI = 0$ implies $a + d = 0$ and $b = c$. The condition $J^2 = 3\mathrm{Id}$ then becomes $(a^2 + b^2)\mathrm{Id} = 3\mathrm{Id}$. It therefore suffices to observe that $a = 1$, $b = 3$ satisfy the equation $a^2 + b^2 = 3$ to conclude.

**Exercise 10.2.7** Let $k$ be a field of odd characteristic.

(i) Show that, for any $a \in k^\times$, the quaternion algebra $(a,1)_k$ is split.

(ii) Deduce that, if $b \in k^\times$ is a norm from $k(\sqrt{a})$, then the algebra $(a,b)_k$ is split.
(iii) Deduce further that, for $a \neq 0, 1$, the algebra $(a, 1-a)_k$ is split.

A class of central simple algebras generalising the quaternion algebras is the class of cyclic algebras.

**Definition 10.2.8** Let $\ell/k$ be a finite cyclic extension of fields of degree $n$. Given $b \in k^*$ and a generator $\sigma$ of $\mathrm{Gal}(\ell/k)$, the *cyclic algebra* $(\ell/k, \sigma, b)$ is defined as follows: let $\ell[x]_\sigma$ denote the non-commutative "twisted polynomial ring", where $ax = x\sigma(a)$ for all $a \in \ell$; then $(\ell/k, \sigma, b)$ is the quotient of $\ell[x]_\sigma$ by the two-sided ideal $(x^n - b)$.

More explicitly, the $k$-algebra $(\ell/k, \sigma, b)$ is the vector space over $\ell$ with basis $1, x, x^2, \dots, x^{n-1}$, and multiplication defined by

$$x^i x^j = \begin{cases} x^{i+j} & \text{if } i+j < n; \\ bx^{i+j-n} & \text{if } i+j \geq n; \end{cases}$$
$$ax = x\sigma(a) \quad \text{for all } a \in \ell.$$

**Exercise 10.2.9** Verify that, when $k$ has odd characteristic, $\ell = k(\sqrt{a})$ is a quadratic extension of $k$, and $\sigma$ is the non-trivial element of $\mathrm{Gal}(\ell/k)$, the cyclic algebra $(\ell/k, \sigma, b)$ is isomorphic to the quaternion algebra $(a,b)_k$ defined above.

We shall see later **?**, using Galois cohomology, that the algebra $(\ell/k, \sigma, f)$ is split if and only if $b$ is a norm for the extension $\ell/k$, thus extending the case of quaternion algebras.

If $k$ contains a primitive $n$th root of unity $\zeta$, then Kummer theory shows that every cyclic extension $\ell/k$ of degree $n$ is of the form $\ell = k(\sqrt[n]{a})$ for some $a \in k^\times$. Given $a, b \in k^\times$, we let $\sigma \in \mathrm{Gal}(k(\sqrt[n]{a})/k)$ be the automorphism that maps $\sqrt[n]{a} \mapsto \zeta\sqrt[n]{a}$, and define the cyclic algebra $(a,b)_{k,\zeta}$ by

$$(a,b)_{k,\zeta} = (k(\sqrt[n]{a})/k, \sigma, b).$$

We now turn to classifying central simple algebras over a field. An important step is the following theorem of Wedderburn, which reduces the problem to that of understanding division algebras.

**Theorem 10.2.10** (Wedderburn) *Let $A$ be a central simple algebra over a field $k$. There are a division algebra $D$, unique up to isomorphism, and a positive integer $n$ such that $A$ is isomorphic to* $\mathrm{M}_n(D)$.

*Proof* See Gille and Szamuely (2006, Theorem 2.1.3). □

The division algebra $D$ of Theorem 10.2.10 is necessarily finite-dimensional and central over $k$. For quaternion algebras, Wedderburn's theorem has a very simple consequence: a quaternion algebra over a field $k$ is either a division algebra or is isomorphic to $\mathrm{M}_2(k)$.

Next we examine tensor products of central simple algebras.

**Lemma 10.2.11** *Let $k$ be a field and let $A$ and $B$ be finite-dimensional simple algebras over $k$. If either $A$ or $B$ is central over $k$, then the tensor product $A \otimes_k B$ is simple.*

*Proof* See Milne (2008, Proposition IV.2.6). □

For a ring $R$, the symbol $Z(R)$ denotes the centre of $R$.

**Lemma 10.2.12** *Let $A$ and $B$ be two finite-dimensional algebras over a field $k$. Then, inside $A \otimes_k B$, the equality $Z(A \otimes_k B) = Z(A) \otimes_k Z(B)$ holds.*

*Proof* This is a special case of Milne (2008, Proposition IV.2.3). □

**Lemma 10.2.13** *Let $A$ and $B$ be central simple algebras over $k$. The $k$-algebra $A \otimes_k B$ is also a central simple algebra over $k$.*

*Proof* This follows immediately from Lemmas 10.2.11 and 10.2.12. □

Given any $k$-algebra $A$, we denote by $A^{\mathrm{opp}}$ the *opposite algebra* of $A$: $A^{\mathrm{opp}}$ is the $k$-algebra whose underlying vector space is the same as the vector space underlying $A$, and on which multiplication is defined by $a \cdot_{A^{\mathrm{opp}}} b = b \cdot_A a$, that is, multiplication in $A^{\mathrm{opp}}$ is multiplication in $A$ in the opposite order. If $A$ is a central simple algebra over $k$ of dimension $n$, then $A^{\mathrm{opp}}$ is also a central simple algebra over $k$ of dimension $n$.

We next give a characterisation of central simple algebras in terms of their endomorphism rings.

**Proposition 10.2.14** *Let $A$ be a non-zero algebra over $k$ of finite dimension. Then $A$ is a central simple algebra over $k$ if and only if the homomorphism of $k$-algebras*

$$\begin{aligned} \phi : A \otimes_k A^{\mathrm{opp}} &\longrightarrow \mathrm{End}_k(A) \\ a \otimes b &\longmapsto (x \mapsto axb) \end{aligned}$$

*is an isomorphism.*

*Proof* Suppose that $A$ is a central simple algebra over $k$. By Lemma 10.2.13 the algebra $A \otimes_k A^{\mathrm{opp}}$ is also a central simple algebra. In particular, the homomorphism $\phi$ is either injective or zero. Since the identity element on the left maps to the identity element on the right, $\phi$ is injective. Comparing dimensions shows that $\phi$ is also surjective, hence an isomorphism.

In order to show that $\phi$ being an isomorphism implies that $A$ is central, suppose on the contrary that $a$ is an element of the centre of $A$ that does not lie in $k$. Then $a \otimes 1 - 1 \otimes a$ is a non-zero element of the kernel of $\phi$, and so $\phi$ is not an isomorphism.

Finally, suppose that $\phi$ is an isomorphism, and let us show that $A$ is simple. Let $I$ be a two-sided ideal in $A$. Let $n$ be the dimension of $A$ over $k$, and let $i$ be the dimension of $I$. Choosing a complementary subspace $V$ to $I$, the tensor product $A \otimes A^{\mathrm{opp}}$ is the direct sum of the four subspaces $I \otimes_k I$, $I \otimes_k V$, $V \otimes_k I$ and $V \otimes_k V$. The first three of these are mapped by $\phi$ into the subspace $\mathrm{Hom}_k(A, I) \subset \mathrm{End}_k(A)$. Since $\phi$ is injective, looking at dimensions gives

$$i^2 + i(n-i) + (n-i)i \leq ni$$

and so we have either $i = 0$, or $2n - i \leq n$ and therefore $i = n$. Thus $I$ is either the zero ideal or the whole of $A$, showing that $A$ is simple. □

We conclude this section with a discussion of the reduced norm. The determinant of a matrix is a tool that is useful in many ways, one of which is to characterise whether the matrix is invertible. On a general central simple algebra, the rôle of the determinant is played by the reduced norm, which we will now define.

Let $A$ be a central simple algebra over a field $k$. The *norm* of an element $a \in A$, denoted $\mathrm{N}_{A/k}(a)$, is the determinant of the linear endomorphism $x \mapsto ax$ of $A$. (This is equal to the determinant of the endomorphism $x \mapsto xa$.) It is clear that, for all $a, b \in A$, we have $\mathrm{N}_{A/k}(ab) = \mathrm{N}_{A/k}(a)\mathrm{N}_{A/k}(b)$. It also follows easily from the definition that, if we fix a basis of $A$, then $\mathrm{N}_{A/k}(a)$ is a homogeneous polynomial of degree $\dim_k A$ in the coordinates of $a$, with coefficients in $k$.

Let us consider the case when $A = \mathrm{M}_n(k)$ is a matrix algebra. As a vector space, $A$ is the direct sum of the $n$ subspaces $E_1, \dots, E_n$, where $E_i \cong k^n$ consists of those matrices whose entries outside the $i$th column are zero. Multiplication on the left by any matrix preserves this decomposition. For any matrix $m \in \mathrm{M}_n(k)$, the action of $m$ on $E_i$ is the same as its natural action on $k^n$, so we see $\mathrm{N}_{A/k}(m) = \big(\det(m)\big)^n$.

**Definition 10.2.15** Let $A$ be a central simple algebra over a field $k$, and fix an isomorphism $\phi : A \otimes_k k^{\mathrm{sep}} \to \mathrm{M}_n(k^{\mathrm{sep}})$ of $k^{\mathrm{sep}}$-algebras. The *reduced norm* of an element $a \in A$ is defined to be $\mathrm{Nrd}_{A/k}(a) = \det(\phi(a \otimes 1))$.

*Remark* 10.2.16 (i) The definition of $\mathrm{Nrd}_{A/k}(a)$ does not depend on the choice of the isomorphism $\phi$ since, by Corollary 10.3.13, any two such isomorphisms are related by conjugation by an element of $\mathrm{M}_n(k^{\mathrm{sep}})$, which does not affect the determinant.

(ii) We have $\big(\mathrm{Nrd}_{A/k}(a)\big)^n = \mathrm{N}_{A/k}(a)$, where $n = \sqrt{\dim_k A}$. Indeed, the determinant of a linear transformation is unaffected by changing the base field, so this follows from the case of matrix algebras, described above.

(iii) It follows immediately from the definition that, for all $a, b \in A$, we have $\mathrm{Nrd}_{A/k}(ab) = \mathrm{Nrd}_{A/k}(a)\,\mathrm{Nrd}_{A/k}(b)$.

Like the determinant, the reduced norm is a homogeneous polynomial function on an algebra $A$:

**Lemma 10.2.17** *Let $A$ be a central simple algebra of dimension $n^2$ over a field $k$. Fix a basis $a_1, \dots, a_{n^2}$ for $A$ over $k$. There is a homogeneous polynomial $F$ of degree $n$ in $k[x_1, \dots, x_{n^2}]$ such that, for all $\lambda_1, \dots, \lambda_{n^2}$ in $k$, we have*

$$\mathrm{Nrd}_{A/k}(\lambda_1 a_1 + \cdots + \lambda_{n^2} a_{n^2}) = F(\lambda_1, \dots, \lambda_{n^2}).$$

*Proof* Fix an isomorphism $\phi\colon A \otimes_k k^{\mathrm{sep}} \to \mathrm{M}_n(k^{\mathrm{sep}})$. Let $a = \sum_{i=1}^{n^2} \lambda_i a_i$ be an element of $A$. Since $\phi$ is a $k^{\mathrm{sep}}$-linear map, the entries of the matrix $\phi(a \otimes 1)$ are linear polynomials in the $\lambda_i$, with coefficients in $k^{\mathrm{sep}}$. Therefore the determinant $\mathrm{Nrd}_{A/k}(a) = \det(\phi(a \otimes 1))$ is given by $F(\lambda_1, \dots, \lambda_{n^2})$, where $F$ is a homogeneous polynomial of degree $n$ with coefficients in $k^{\mathrm{sep}}$. We must shows that $F$ actually has coefficients in $k$.

The polynomial $F^n$ has coefficients in $k$, since $F^n$ is the polynomial expressing $\mathrm{N}_{A/k}(a)$ in terms of the coordinates $\lambda_i$. Let $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$ be an automorphism, and let $\sigma F$ be the polynomial obtained by letting $\sigma$ act on the coefficients of $F$; it satisfies $(\sigma F)(\mathbf{x}) = \sigma(F(\sigma^{-1}\mathbf{x}))$ for all $\mathbf{x} \in (k^{\mathrm{sep}})^{n^2}$. Then we have $(\sigma F)^n = \sigma(F^n) = F^n$, and therefore $\sigma F = \zeta F$ where $\zeta$ is an $n$-th root of unity in $k^{\mathrm{sep}}$. To show $\zeta = 1$, it is enough to exhibit a vector $\mathbf{x} \in k^{n^2}$ such that $F(\mathbf{x})$ lies in $k$, since for such an $\mathbf{x}$ we have $\zeta F(\mathbf{x}) = (\sigma F)(\mathbf{x}) = F(\mathbf{x})$. But $\phi$ is an isomorphism of algebras, so we have $\mathrm{Nrd}(1_A) = \det(\phi(1)) = \det(\mathrm{Id}) = 1$. Writing $1_A = \sum_i \lambda_i a_i$ gives $F(\lambda_1, \dots, \lambda_n) = 1$. Thus $F$ is fixed by $\sigma$. This holds for all $\sigma \in \mathrm{Gal}(k^{\mathrm{sep}}/k)$, so $F$ has coefficients in $k$. □

## 10.3 The Brauer group of a field

**Definition 10.3.1** Let $k$ be a field. Two central simple algebras $A$ and $B$ over $k$ are *equivalent* if there are positive integers $m$ and $n$ such that $\mathrm{M}_m(A)$ and $\mathrm{M}_n(B)$ are isomorphic. This induces an equivalence relation on the set of isomorphism classes of central simple algebras over $k$, with transitivity following from Exercise 10.2.2(i). We denote the equivalence class of the central simple algebra $A$ over $k$ by $[A]$, and call it the *Brauer class* of $A$.

Wedderburn's Theorem shows that two central simple algebras are equivalent if and only if they are both matrix algebras over the same division algebra.

We are now in a position to define the Brauer group of the field $k$.

**Definition 10.3.2** The *Brauer group* $\mathrm{Br}(k)$ of the field $k$ is the abelian group whose elements are the equivalence classes of central simple algebras over $k$, with group operation $[A] \cdot [B] := [A \otimes_k B]$.

The Brauer group is indeed an abelian group. Associativity and commutativity come from standard properties of the tensor product. The identity in $\mathrm{Br}(k)$ is the class of the field $k$. By Proposition 10.2.14, the inverse of the class of a central simple algebra $A$ is the class of $A^{\mathrm{opp}}$, because $\mathrm{End}_k(A)$ is isomorphic to a matrix algebra over $k$.

Wedderburn's Theorem shows that each class in the Brauer group of a field contains a unique division algebra; the other algebras in that class are isomorphic to matrix algebras over that division algebra.

As an example of multiplication in the Brauer group, we look at the tensor product of two quaternion algebras. In general, there is no reason to expect the tensor product of two quaternion algebras to be equivalent to a third quaternion algebra, but we have the following bilinearity result.

**Lemma 10.3.3** *Let $k$ be a field of odd characteristic and let $a, b, c$ be non-zero elements of $k$. Then there is an isomorphism*

$$(a,b)_k \otimes_k (a,c)_k \cong \mathrm{M}_2((a,bc)_k).$$

*Proof* This can be proved by exhibiting an explicit isomorphism: see Gille and Szamuely (2006, Lemma 1.5.2). □

In other words, the product of the classes $[(a,b)_k]$ and $[(a,c)_k]$ in $\mathrm{Br}\,k$ is the class $[(a,bc)_k]$. So the map from $k^\times \times k^\times$ to $\mathrm{Br}\,k$ that sends $(a,b)$ to the class of the algebra $(a,b)_k$ is a bilinear map of abelian groups. (We have only proved linearity on one side, but it is also symmetric.) Combined with Exercise 10.2.5(ii), this also shows that the class of a quaternion algebra has order dividing 2 in the Brauer group.

*Remark* 10.3.4 Let $k$ be any field. The second *Milnor K-group* of $k$, written $\mathrm{K}_2(k)$, is defined to be the quotient of the group $k^\times \otimes_{\mathbf{Z}} k^\times$ by the subgroup generated by all elements of the form $a \otimes (1-a)$ for $a \in k$ with $a \neq 0, 1$. If $k$ has odd characteristic then Lemma 10.3.3, together with Exercises 10.2.5(ii) and 10.2.7, shows that the map sending $a \otimes b$ to the class of $(a,b)_k$ induces a homomorphism $\mathrm{K}_2(k) \to \mathrm{Br}\, k[2]$. In fact, the induced homomorphism $\mathrm{K}_2(k)/2\mathrm{K}_2(k) \to \mathrm{Br}\, k[2]$ is an *isomorphism*, as was proved by Merkurjev (1981). This means that any class in $\mathrm{Br}\, k[2]$ is represented by a tensor product of quaternion algebras.

More generally, if $k$ contains a primitive $n$th root of unity $\zeta$, then there is a similar homomorphism $\mathrm{K}_2(k)/n\mathrm{K}_2(k) \to \mathrm{Br}\, k[n]$ defined by sending the class of $a \otimes b$ to the class of the cyclic algebra $(a,b)_{k,\zeta}$. That this is also an isomorphism is a celebrated theorem of Merkurjev and Suslin (1982).

**Exercise 10.3.5** Prove that if $k$ is algebraically closed, then $\mathrm{Br}(k)$ consists of the single element $[k]$. (Hint: it suffices to show that there are no non-trivial division algebras over $k$. If $D$ is a division algebra over $k$, let $d$ be an element of $D$ and consider $k(d)$.)

In fact, a stronger statement is true.

**Proposition 10.3.6** (Noether, Köthe) *Let $k$ be a separably closed field. The Brauer group $\mathrm{Br}\, k$ is trivial.*

*Proof* See Bourbaki (2012, §13, Proposition 3). □

**Exercise 10.3.7** Find an explicit isomorphism between $\mathbb{H}_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathrm{M}_2(\mathbf{C})$.

We now look at the behaviour of central simple algebras under base extension.

**Lemma 10.3.8** *Let $\ell/k$ be a field extension and $A$ a finite-dimensional algebra over $k$. The algebra $A$ is a central simple algebra over $k$ if and only if $A \otimes_k \ell$ is a central simple algebra over $\ell$.*

*Proof* Let $\phi\colon A \otimes_k A^{\mathrm{opp}} \to \mathrm{End}_k(A)$ be the homomorphism of Proposition 10.2.14. Let $A_\ell$ denote the base change $A \otimes_k \ell$. Using the natural isomorphisms

$$(A \otimes_k A^{\mathrm{opp}}) \otimes_k \ell \cong A_\ell \otimes_\ell A_\ell^{\mathrm{opp}} \quad \text{and} \quad \mathrm{End}_k(A) \otimes_k \ell \cong \mathrm{End}_\ell(A_\ell)$$

identifies the analogous homomorphism $\phi_\ell\colon A_\ell \otimes_\ell A_\ell^{\mathrm{opp}} \to \mathrm{End}_\ell(A_\ell)$ with the base change $\phi \otimes 1$. The linear map $\phi$ is an isomorphism if and only if its base change $\phi_\ell$ is an isomorphism, so Proposition 10.2.14 gives the desired result. □

We call $A \otimes_k \ell$ the *extension* of $A$ to $\ell$. It is easy to check that extension to $\ell$ defines a group homomorphism $\mathrm{Br}(k) \to \mathrm{Br}(\ell)$, making Br into a functor from the category of fields to the category of abelian groups.

Define $\mathrm{Br}(\ell/k)$ to be the kernel of the extension map $\mathrm{Br}\, k \to \mathrm{Br}\, \ell$.

**Proposition 10.3.9** *Let $k$ be a field and let $A$ be a finite-dimensional algebra over $k$. The algebra $A$ is a central simple algebra over $k$ if and only if there is a finite separable extension $\ell/k$ such that $A \otimes_k \ell$ is isomorphic to a matrix algebra.*

*Proof* Assume that there is such an extension $\ell$. In particular, $A \otimes_k \ell$ is a central simple algebra, so Lemma 10.3.8 shows that $A$ is a central simple algebra.

Conversely, suppose that $A$ is a central simple algebra over $k$. Let $k^{\text{sep}}$ be a separable closure of $k$. By Proposition 10.3.6, there is an isomorphism $\phi\colon A \otimes_k k^{\text{sep}} \to \mathrm{M}_n(k^{\text{sep}})$ for some positive integer $n$. Choose a basis $a_1, \ldots, a_r$ for $A$ over $k$, and let $\ell$ be the subextension of $k^{\text{sep}}/k$ generated by the entries of the matrices $\phi(a_1 \otimes 1), \ldots, \phi(a_r \otimes 1)$. Then the restriction of $\phi$ to $A \otimes_k \ell$ induces an isomorphism of $A \otimes_k \ell$ with $\mathrm{M}_n(\ell)$. □

A field $\ell$ as in the proposition is called a *splitting field* for $A$, and we say that $A$ *splits* over $\ell$.

The following two corollaries are immediate.

**Corollary 10.3.10** *Let $k$ be a field. The Brauer group* $\mathrm{Br}\, k$ *is the union of the groups* $\mathrm{Br}(\ell/k)$ *as $\ell$ runs over all finite separable extensions of $k$.* □

**Corollary 10.3.11** *The dimension of a central simple algebra over $k$ is a square.* □

Next we state the Skolem–Noether theorem, which has the consequence that all automorphisms of a central simple algebra are inner. We will need this later when we give the cohomological description of the Brauer group.

**Theorem 10.3.12** (Skolem–Noether) *Let $k$ be a field, let $A$ be a simple $k$-algebra, let $B$ be a central simple $k$-algebra and let $f, g\colon A \to B$ be two homomorphisms. Then there exists an invertible element $b \in B$ such that $f(x) = bg(x)b^{-1}$ for all $x \in A$.*

*Proof* See Milne (2008, Theorem IV.2.10). □

**Corollary 10.3.13** *Let $A$ be a central simple algebra over a field $k$. Then every automorphism of $A$ is inner, that is, given by conjugation by an element of $A$.*

*Proof* If $\phi\colon A \to A$ is an automorphism, then apply Theorem 10.3.12 with $f = \phi$ and $g = \mathrm{id}_A$. □

## 10.4 Brauer groups of some fields

In this section we describe the Brauer groups of several important fields, including the real numbers, the $p$-adic numbers and the rational numbers. We begin with the real numbers.

**Theorem 10.4.1** (Frobenius) *If $A$ is a finite-dimensional division algebra over the real numbers $\mathbf{R}$, then $A$ is isomorphic to either $\mathbf{R}$ itself, the field $\mathbf{C}$ of complex numbers, or the algebra $\mathbb{H}_{\mathbf{R}}$ of Hamilton quaternions.*

*Proof* See Bourbaki (2012, Section 19, Théorème 1). □

**Corollary 10.4.2** *The Brauer group of $\mathbf{R}$ has order 2, generated by the class consisting of matrix algebras over the quaternion algebra $\mathbb{H}_{\mathbf{R}}$.*

We will be able to prove for ourselves that $\operatorname{Br}\mathbf{R}$ has order 2, without using Frobenius' theorem, after developing Galois cohomology in Chapter 14.

Next, we turn to local fields.

**Theorem 10.4.3** *Let $k$ be a number field and let $v$ be a finite place of $k$. There is a canonical isomorphism $\operatorname{inv}_v\colon \operatorname{Br} k_v \to \mathbf{Q}/\mathbf{Z}$.*

*Proof* We will prove this in Section 15.2 using Galois cohomology. For a direct construction of $\operatorname{inv}_v$, see Milne (2008, Section IV.4). □

The isomorphism $\operatorname{inv}_v$ in the preceding theorem is called the *Hasse invariant map* at $v$.

**Exercise 10.4.4** Let $v$ be a valuation of $\mathbf{Q}$. Show that the value of the invariant at $v$ of the quaternion algebra $(a,b)_{\mathbf{Q}}$ is equal to the Hilbert symbol $(a,b)_v$ (under the unique group isomorphism between $\{1,-1\}$ and $\frac{1}{2}\mathbf{Z}/\mathbf{Z}$).

To state the following fundamental theorem, describing the Brauer group of a number field $k$, we also need to define invariant maps at infinite places. If $v$ is a real place of $k$, then define $\operatorname{inv}_v\colon \operatorname{Br} k_v \to \mathbf{Q}/\mathbf{Z}$ to be the unique injective homomorphism mapping $\operatorname{Br} k_v \cong \operatorname{Br}\mathbf{R}$ to the subgroup $\{0, \frac{1}{2}\} \subset \mathbf{Q}/\mathbf{Z}$. If $v$ is a complex place of $k$, then $\operatorname{Br} k_v$ is trivial and we define $\operatorname{inv}_v$ to be the zero map. Recall that $\Omega_k$ denotes the set of all places of a number field $k$. The following theorem is closely related to the reciprocity theorem of global class field theory, and generalises the product formula for the Hilbert symbol.

**Theorem 10.4.5** *Let $k$ be a number field. There is an exact sequence*

$$0 \to \operatorname{Br} k \to \bigoplus_{v \in \Omega_k} \operatorname{Br} k_v \xrightarrow{\sum_v \operatorname{inv}_v} \mathbf{Q}/\mathbf{Z} \to 0 \qquad (10.1)$$

*where the map* $\operatorname{Br} k \to \bigoplus \operatorname{Br} k_v$ *is the diagonal map induced by the inclusions of $k$ into each completion $k_v$.*

*Proof* For a proof using cohomology, see Milne (2008, VIII.4). □

The injectivity of the map $\operatorname{Br} k \to \bigoplus \operatorname{Br} k_v$ is the celebrated theorem of Albert and Brauer–Hasse–Noether; see Roquette (2005) for a historical discussion of this theorem and its proof.

The sequence (10.1) shows a very important property: the local invariants of a Brauer class satisfy a global relation. It is this compatibility condition which will form the basis of the Brauer–Manin obstruction.

We conclude this section by showing that a certain important class of fields, namely the quasi-algebraically closed fields, have trivial Brauer group. In particular, this proves that the Brauer group of a finite field is trivial.

**Definition 10.4.6** A field $k$ is said to be *quasi-algebraically closed*, or $C_1$, if every non-constant homogenous polynomial of degree $d$ in $n$ variables over $k$ satisfying $n > d$ has a non-trivial zero in $k^n$.

**Example 10.4.7** The Chevalley–Warning theorem (Theorem 2.2.4) shows that a finite field is $C_1$.

**Theorem 10.4.8** (Tsen) *Let $K$ be the function field of a curve over an algebraically closed base field $k$. Then $K$ is a $C_1$ field.*

*Proof* See Greenberg (1969, Theorem 3.6), Shatz (1972, Theorem IV.3.24) or Kollár (1996, Theorem IV.6.5). □

A property of $C_1$ fields that is important to us is that they have trivial Brauer group.

**Theorem 10.4.9** *Let $k$ be a $C_1$ field. Then* $\operatorname{Br} k$ *is trivial.*

*Proof* It is enough to show that no central simple algebra of dimension $n^2 > 1$ over $k$ is a division algebra. Let $A$ be such an algebra over $k$, and choose a basis $a_1, \dots, a_{n^2}$ for $A$. By Lemma 10.2.17, we have

$$\operatorname{Nrd}_{A/k}(\lambda_1 a_1 + \cdots + \lambda_{n^2} a_{n^2}) = F(\lambda_1, \dots, \lambda_{n^2}).$$

for some homogeneous $F \in k[x_1, \dots, x_{n^2}]$ of degree $n$. Since $k$ is $C_1$, the polynomial $F$ admits a non-trivial zero. Therefore there is a non-zero element $a \in A$ satisfying $\operatorname{Nrd}_{A/k}(a) = 0$. By multiplicativity of the reduced norm, $a$ cannot be invertible; thus $A$ is not a division algebra. □

**Corollary 10.4.10** (i) *If $k$ is a finite field, then* $\operatorname{Br} k$ *is trivial.*
(ii) *If $C$ is a curve over an algebraically closed field, then* $\operatorname{Br} \kappa(C)$ *is trivial.*

*Proof* These follow from Example 10.4.7 and Theorem 10.4.8, respectively. □

## 10.5 Motivation for the Brauer group of a variety

We conclude this chapter with a look at the Brauer group of the function field of a variety, which leads us to want to define the Brauer group of the variety. That will be the goal of the following two chapters.

Let $X$ be a smooth, geometrically irreducibe, projective variety over a field $k$. The Brauer group of the field $\kappa(X)$ consists of equivalence classes of central simple algebras over $\kappa(X)$. Rather like evaluating an element of $\kappa(X)$ at a point of $X$, we can ask what it might mean to evaluate an element of $\operatorname{Br}\kappa(X)$ at a point of $X$.

**Example 10.5.1** Take $X = \mathbf{A}^1_{\mathbf{Q}}$. The function field of $X$ is $\kappa(X) = \mathbf{Q}(t)$. Consider the quaternion algebra $\mathscr{A} = (-1,t)_{\mathbf{Q}(t)}$. For any $a \in \mathbf{Q}$, we can try to evaluate $\mathscr{A}$ at the point defined by $t = a$, simply by substituting $t = a$ into the definition of $\mathscr{A}$. If $a$ is non-zero, this gives us the quaternion algebra $(-1,a)_{\mathbf{Q}}$, which defines a class in $\operatorname{Br}\mathbf{Q}$. However, if $a$ is zero, then we obtain an algebra $(-1,0)_{\mathbf{Q}}$ that is not a central simple algebra over $\mathbf{Q}$.

There is no need to restrict ourselves to $a \in \mathbf{Q}$. For any extension $K/\mathbf{Q}$ and any non-zero $a \in K$, we can evaluate $\mathscr{A}$ at $t = a$ to obtain an element of $\operatorname{Br}K$. So this kind of "evaluation" works for all points in the open set $U$ defined by $t \neq 0$. In fact, it will turn out that the class of $\mathscr{A}$ in $\operatorname{Br}\kappa(X)$ lies in the subgroup $\operatorname{Br}U$.

**Example 10.5.2** For a more complicated example, let $X$ be the elliptic curve over $\mathbf{Q}$ defined by the Weierstrass equation

$$y^2 = (x-e_1)(x-e_2)(x-e_3)$$

and let $\mathscr{A}$ be the quaternion algebra $(3, x-e_1)_{\kappa(X)}$. Evaluating at any point of $X(\mathbf{Q})$ apart from $(e_1,0)$ gives a well-defined quaternion algebra over $\mathbf{Q}$, but it looks as if we cannot evaluate $\mathscr{A}$ at the point $(e_1,0)$. However, dividing $x-e_1$ by $y^2$ shows (as in Exercise 10.2.5(ii)) that $\mathscr{A}$ is isomorphic to the algebra $(3,((x-e_2)(x-e_3))^{-1})_{\kappa(X)}$. Here we can indeed substitute $(e_1,0)$ to get an algebra over $\mathbf{Q}$. So it seems that, just as with rational functions on varieties, it is sometimes possible to extend the domain of definition by writing the algebra in a different way. In fact, the algebra $\mathscr{A}$ lies in the subgroup $\operatorname{Br}X \subset \operatorname{Br}\kappa(X)$.

Following the analogy with rational functions, it is of course not surprising that not every rational function $f \in \kappa(X)$ can be evaluated at every point of $X$. If that were true, then a $k$-point of $X$ would give a non-zero evaluation homomorphism $\kappa(X) \to k$, and there are no such homomorphisms. The way to understand this is to introduce the local ring at a point $P$ of $X$. The local ring $\mathscr{O}_{X,P}$ is a subring of the function field $\kappa(X)$, and it admits an evaluation homomorphism $\mathscr{O}_{X,P} \to \kappa(P)$. The functions lying in $\mathscr{O}_{X,P}$ are precisely those for which "evaluation at $P$" makes sense.

Exactly the same approach can be taken with Brauer groups. Rather than hoping for evaluation homomorphisms $\operatorname{Br} \kappa(X) \to \operatorname{Br} k$, we will define the Brauer group of the local ring $\mathscr{O}_{X,P}$ at a point $P$. This fits into a diagram

$$\operatorname{Br} \kappa(X) \leftarrow \operatorname{Br} \mathscr{O}_{X,P} \to \operatorname{Br} \kappa(P).$$

Thus any element of $\operatorname{Br} \kappa(X)$ lying in the image of $\operatorname{Br} \mathscr{O}_{X,P}$ can be evaluated at $P$ to obtain an element of $\operatorname{Br} \kappa(P)$. In fact we shall see that, since $X$ is smooth, the map $\operatorname{Br} \mathscr{O}_{X,P} \to \operatorname{Br} \kappa(X)$ is injective, and so we can consider $\operatorname{Br} \mathscr{O}_{X,P}$ as a subgroup of $\operatorname{Br} \kappa(X)$. It consists of those elements of $\operatorname{Br} \kappa(X)$ that "can be evaluated at $P$".

A geometrically irreducible projective variety admits no non-constant regular functions: there are no non-constant elements of $\kappa(X)$ that lie in $\mathscr{O}_{X,P}$ for all points $P$. For Brauer groups, however, this is more interesting: depending on the geometry of the variety, there can indeed be non-constant elements in $\operatorname{Br} \kappa(X)$ that lie in $\operatorname{Br} O_{X,P}$ for *all* points $P$ of $X$. Such elements constitute the Brauer group of $X$.

In Chapter 11 we will start to make the above discussion precise by defining the Brauer group of a ring, and in particular the Brauer group of a local ring. In Chapter 12 we will use this to define the Brauer group of a variety.

# 11

# The Brauer group of a ring

The definition of the Brauer group of a field can be generalised to apply to arbitrary commutative rings. We will need this, at least in the case of local rings, in order to define the Brauer group of a variety in the next chapter.

Central simple algebras over fields are examples of a more general class of algebras over commutative rings, called *Azumaya algebras*. Recall that a central simple algebra over a field is a finite-dimensional vector space, endowed with the structure of an algebra, and satisfying certain properties. If the base field is to be replaced by a commutative ring, then a natural generalisation is to replace vector spaces with projective modules, which are not necessarily free but are *locally* free. This gives a theory of Brauer groups closely matching that over fields. Similarly, over a field, we took the trivial class in the Brauer group to consist of matrix algebras. A matrix algebra over a field is the endomorphism algebra of a vector space, and so when we define the Brauer group of a ring we will take the trivial class to consist of the endomorphism rings of projective modules.

The study of Brauer groups of general commutative rings goes back to Auslander and Goldman (1960), building on the definition of "maximally central algebras" of Azumaya (1951). For a useful discussion of various equivalent definitions of Azumaya algebras and their history, see Bass (1967); DeMeyer and Ingraham (1971); Millar (2010).



## 11.1 Some commutative algebra

We begin by recalling some properties of localisation and of finitely generated projective modules. For any prime ideal $\mathfrak{p}$ of a commutative ring $R$ and for any $R$-algebra $M$, let $M_{\mathfrak{p}}$ denote $M \otimes_R R_{\mathfrak{p}}$, let $k(\mathfrak{p})$ denote the field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and let $M(\mathfrak{p})$ denote $M \otimes_R k(\mathfrak{p})$.

**Lemma 11.1.1** *Let $R$ be a commutative ring, and let $f\colon M \to N$ be a homomorphism of $R$-modules. The following are equivalent:*

(i) *$f$ is injective;*
(ii) *for every prime ideal $\mathfrak{p} \subset R$, the induced homomorphism $f_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective;*
(iii) *for every maximal ideal $\mathfrak{m} \subset R$, the induced homomorphism $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective.*

*The same holds with 'injective' replaced by 'surjective'.*

*Proof* See Atiyah and Macdonald (1969, Proposition 3.9). □

Let $R$ be a commutative ring. The algebras that appear in the definition of the Brauer group of $R$ will be finitely generated projective modules over $R$, and we now recall some properties of such modules. A good reference for this is Bourbaki (1998, Section II.5). If a module $M$ over $R$ is finitely generated and projective then, for every prime ideal $\mathfrak{p} \subset R$, the localisation $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$, of finite rank (see Bourbaki, 1998, Section II.5.2, Théorème 1). This allows us to define the *rank* of a projective module $M$ at a prime ideal $\mathfrak{p} \subset R$ (or, more succinctly, the $\mathfrak{p}$-rank of $M$) to be the rank of the free module $M_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. (This is equal to the dimension of the vector space $M(\mathfrak{p})$ over the field $k(\mathfrak{p})$.) Since the localisation of a free module is again a free module of the same rank, it follows that, if $\mathfrak{p} \subset \mathfrak{q}$ are two prime ideals, then the $\mathfrak{q}$-rank of $M$ is equal to the $\mathfrak{p}$-rank. In particular, if $R$ is an integral domain, then every prime ideal contains the prime ideal 0, and so the $\mathfrak{p}$-rank of $M$ is the same for every prime ideal $\mathfrak{p}$.

Recall that an $R$-module $M$ is called *faithful* if the natural map $R \to \operatorname{End}_R(M)$ is injective.

**Lemma 11.1.2** *Let $M$ be finitely generated projective module over a commutative ring $R$. The following are equivalent:*

(i) *$M$ is a faithful $R$-module;*
(ii) *for every prime ideal $\mathfrak{p} \subset R$, the $\mathfrak{p}$-rank of $M$ is non-zero;*
(iii) *for every maximal ideal $\mathfrak{m} \subset R$, the $\mathfrak{m}$-rank of $M$ is non-zero.*

*If $R$ is an integral domain, then these are also equivalent to: $M$ is non-zero.*

*Proof* For a prime ideal $\mathfrak{p} \subset R$, the natural map $R_{\mathfrak{p}} \to \operatorname{End}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ identifies $R_{\mathfrak{p}}$ with the scalar $n \times n$ matrices, where $n$ is the $\mathfrak{p}$-rank of $M$. This map is injective if and only if $n$ is non-zero. Now apply Lemma 11.1.1.

If $R$ is an integral domain, then the $\mathfrak{p}$-rank of $M$ is the same at all prime ideals $\mathfrak{p}$, and is zero if and only if $M$ is zero. □

For finitely generated projective modules, the following lemma makes a useful companion to Lemma 11.1.1.

**Lemma 11.1.3** *Let $R$ be a local ring with maximal ideal $\mathfrak{m}$, and let $f\colon M \to N$ be a homomorphism of free modules of finite rank over $R$. The following are equivalent:*

(i) *$f$ is an isomorphism;*
(ii) *the induced homomorphism $M(\mathfrak{m}) \to N(\mathfrak{m})$ is an isomorphism.*

*Proof* The implication (i)⇒(ii) is immediate; let us prove (ii)⇒(i). By assumption the vector spaces $M(\mathfrak{m})$ and $N(\mathfrak{m})$ have equal dimension, so $M$ and $N$ have equal rank. The determinant of a square matrix representing $f$ is nonzero modulo $\mathfrak{m}$, so is a unit in $R$; therefore the matrix is invertible and $f$ is an isomorphism. □

We now examine the module of homomorphisms between two finitely generated projective modules.

**Lemma 11.1.4** *Let $M,N$ be finitely generated projective modules over a commutative ring $R$. Then the $R$-module $\operatorname{Hom}(M,N)$ is also finitely generated and projective.*

*Proof* By Bourbaki (1998, Section II.5.2, Théorème 1), there exist a finite number of elements $f_1,\dots,f_r \in R$ that generate $R$ as an ideal, and such that, for every $i$, both localisations $M_{f_i}$ and $N_{f_i}$ are free of finite rank over $R_{f_i}$. Therefore the module $\operatorname{Hom}_R(M,N)_{f_i}$, which by Bourbaki (1998, Section II.2.7, Proposition 19) is isomorphic to $\operatorname{Hom}_{R_{f_i}}(M_{f_i},N_{f_i})$, is isomorphic to a module of matrices over $R_{f_i}$ and in particular is free of finite rank. Applying Bourbaki (1998, Section II.5.2, Théorème 1) again shows that $\operatorname{Hom}(M,N)$ is finitely generated and projective over $R$. □

**Lemma 11.1.5** *Let $R \to S$ be a homomorphism of commutative rings and let $M,N$ be finitely generated projective $R$-modules. Then the natural homomorphism of $S$-modules*

$$\operatorname{Hom}_R(M,N) \otimes_R S \to \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S),$$

*is an isomorphism.*

*Proof* If $S$ is a localisation of $R$, then the statement is a special case of Proposition 19 of Bourbaki (1998, Section II.2.7). We will use this to prove the general case by localisation.

Although both sides of the claimed isomorphism are $S$-modules, we can also consider them as $R$-modules, and the map as a homomorphism of $R$-modules.

By Lemma 11.1.1, to see that the homomorphism is an isomorphism, it is enough to check this locally on $R$. Let $\mathfrak{m} \subset R$ be a maximal ideal; then we have $M_\mathfrak{m} \cong R_\mathfrak{m}^m$ and $N_\mathfrak{m} \cong R_\mathfrak{m}^n$ for integers $m, n \geq 0$. It follows that there are isomorphisms

$$\mathrm{Hom}_R(M,N)_\mathfrak{m} \cong \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) \cong \mathrm{M}_{n\times m}(R_\mathfrak{m}).$$

Similarly, using that tensor products commute with localisation (Bourbaki, 1998, Section II.2.7, Proposition 18) we have

$$(M \otimes_R S)_\mathfrak{m} \cong (M_\mathfrak{m} \otimes_{R_\mathfrak{m}} S_\mathfrak{m}) \cong (S_\mathfrak{m})^m$$
$$(N \otimes_R S)_\mathfrak{m} \cong (N_\mathfrak{m} \otimes_{R_\mathfrak{m}} S_\mathfrak{m}) \cong (S_\mathfrak{m})^n$$
$$\mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)_\mathfrak{m} \cong \mathrm{Hom}_{S_\mathfrak{m}}((M \otimes_R S)_\mathfrak{m}, (N \otimes_R S)_\mathfrak{m}) \cong \mathrm{M}_{n\times m}(S_\mathfrak{m}).$$

Under these isomorphisms, the homomorphism

$$(\mathrm{Hom}_R(M,N) \otimes_R S)_\mathfrak{m} \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)_\mathfrak{m}$$

is identified with the natural base change map

$$\mathrm{M}_{n\times m}(R_\mathfrak{m}) \otimes_{R_\mathfrak{m}} S_\mathfrak{m} \to \mathrm{M}_{n\times m}(S_\mathfrak{m}),$$

which is easily seen to be an isomorphism. Applying Lemma 11.1.1 completes the proof. □

## 11.2 Definition of the Brauer group

An *algebra* over a commutative ring $R$ is an $R$-module $A$ endowed with an $R$-linear multiplication $A \times A \to A$, giving $A$ the structure of a ring with identity $1_A$. It follows that the map $R \to A$ sending $r$ to $r.1_A$ is a ring homomorphism whose image is contained in the centre of $A$.

**Lemma 11.2.1** *Let $A$ be an algebra over a commutative ring $R$, and suppose that $A$ is finitely generated and projective as an $R$-module. Then $A$ is faithful if and only if the natural map $R \to A$ is injective.*

*Proof* The map $R \to \mathrm{End}_R A$ factors as $R \to A \to \mathrm{End}_R A$, where the second map sends $a$ to the endomorphism $x \mapsto ax$. This second map is injective because it has the left inverse $f \mapsto f(1)$. Thus $A$ is faithful if and only if $R \to A$ is injective. □

To generalise the notion of central simple algebra to an arbitrary commutative base ring, we use the characterisation of Proposition 10.2.14. For any algebra $A$ over a commutative ring $R$, let $\phi_A$ be the homomorphism of $R$-algebras $A \otimes_R A^{\mathrm{opp}} \to \mathrm{End}_R A$ given by $\phi_A(a \otimes a')(x) = axa'$.

**Definition 11.2.2** Let $R$ be a commutative ring. An *Azumaya algebra over $R$* is an $R$-algebra that is finitely generated, faithful and projective as an $R$-module and for which $\phi_A$ is an isomorphism.

If $R$ is a field, then Proposition 10.2.14 shows that an Azumaya algebra over $R$ is a central simple algebra over $R$.

Let us see how $\phi_A$ behaves under base change. If $S$ is a commutative $R$-algebra, there is the analogous homomorphism $\phi_{A_S}$ for the $S$-algebra $A_S = A \otimes_R S$; it is then easy to check that the diagram

$$\begin{array}{ccc} (A \otimes_R A^{\mathrm{opp}}) \otimes_R S & \xrightarrow{\phi_A \otimes \mathrm{id}_S} & (\mathrm{End}_R A) \otimes_R S \\ \downarrow & & \downarrow \\ A_S \otimes_S A_S^{\mathrm{opp}} & \xrightarrow{\phi_{A_S}} & \mathrm{End}_S A_S \end{array} \tag{11.1}$$

commutes, where the vertical maps are the isomorphisms of **??** and Lemma 11.1.5. This allows us to prove the following characterisation of Azumaya algebras.

**Proposition 11.2.3** *Let $R$ be a commutative ring and $A$ an $R$-algebra that is finitely generated and projective as an $R$-module. The following conditions are equivalent.*

- (i) *$A$ is an Azumaya algebra over $R$.*
- (ii) *For every prime ideal $\mathfrak{p}$ of $R$, the localisation $A_{\mathfrak{p}}$ is an Azumaya algebra over $R_{\mathfrak{p}}$.*
- (iii) *For every prime ideal $\mathfrak{p}$ of $R$, the reduction $A(\mathfrak{p})$ is a central simple algebra over the field $k(\mathfrak{p})$.*
- (iv) *For every maximal ideal $\mathfrak{m}$ of $R$, the localisation $A_{\mathfrak{m}}$ is an Azumaya algebra over $R_{\mathfrak{m}}$.*
- (v) *For every maximal ideal $\mathfrak{m}$ of $R$, the reduction $A(\mathfrak{m})$ is a central simple algebra over the field $k(\mathfrak{m})$.*

*Proof* Lemma 11.1.1, together with the commutativity of (11.1), shows that the following are equivalent: $\phi_A$ is an isomorphism; $\phi_{A_{\mathfrak{p}}}$ is an isomorphism for all $\mathfrak{p}$; and $\phi_{A_{\mathfrak{m}}}$ is an isomorphism for all $\mathfrak{m}$. By Lemma 11.1.4, the ring $\mathrm{End}_R A$ is also a finitely generated, projective $R$-module, and so Lemma 11.1.3 shows that these are also equivalent to: $\phi_{A(\mathfrak{p})}$ is an isomorphism for all $\mathfrak{p}$; and $\phi_{A(\mathfrak{m})}$ is an isomorphism for all $\mathfrak{m}$.

Lemma 11.1.2 shows that the following are equivalent: $A$ is faithful; $A_{\mathfrak{p}}$ is faithful for all $\mathfrak{p}$; $A(\mathfrak{p})$ is non-zero for all $\mathfrak{p}$; $A_{\mathfrak{m}}$ is faithful for all $\mathfrak{m}$; $A(\mathfrak{m})$ is non-zero for all $\mathfrak{m}$.

Combining these two statements proves the proposition. □

An easy consequence is that, for any faithful finitely generated projective $R$-module $M$, the $R$-algebra $A = \mathrm{End}_R(M)$ is an Azumaya algebra over $R$. Indeed, $A$ is finitely generated and projective by Lemma 11.1.4; and, for every maximal ideal $\mathfrak{m}$ of $R$, the reduction $A(\mathfrak{m}) = \mathrm{End}_{k(\mathfrak{m})}(M(\mathfrak{m}))$ is a non-zero matrix algebra.

*Remark* 11.2.4 If $R$ is an integral domain with field of fractions $K$ then Proposition 11.2.3 shows that, for any Azumaya algebra $A$ over $R$, the $K$-algebra $A \otimes_R K$ is a central simple algebra over $K$.

**Definition 11.2.5** Let $R$ be a commutative ring. Two Azumaya algebras $A,B$ over $R$ are *equivalent* if there are finitely generated faithful projective $R$-modules $M,N$ such that the $R$-algebras $A \otimes_R \mathrm{End}_R(M)$ and $B \otimes_R \mathrm{End}_R(N)$ are isomorphic; we denote the equivalence class of $A$ by $[A]$.

Since the tensor product of two central simple algebras is a central simple algebra (Lemma 10.2.13), it follows from Proposition 11.2.3 and the fact that forming tensor products commutes with base change that the tensor product of two Azumaya algebras is again an Azumaya algebra. Observe furthermore that, if $A, A', B, B'$ are Azumaya algebras over a ring $R$ such that $A$ and $A'$ are equivalent and $B$ and $B'$ are also equivalent, then the Azumaya algebras $A \otimes_R B$ and $A' \otimes_R B'$ are equivalent. It is now easy to check that the tensor product induces a group structure on the set of equivalence classes of Azumaya algebras over $R$. Under this operation, the identity is $[R]$, and the inverse of the class of an Azumaya algebra $A$ over $R$ is the class of $A^{\mathrm{opp}}$.

**Definition 11.2.6** Let $R$ be a commutative ring. The *Brauer group* $\mathrm{Br}(R)$ of $R$ is the group of equivalence classes of Azumaya algebras over $R$, with the operation induced by tensor product.

Over a field $K$, the Azumaya algebras over $K$ are exactly the central simple algebras over $K$ and the Brauer groups of Definitions 10.3.2 and 11.2.6 coincide.

**Example 11.2.7** For any commutative ring $R$ and positive integer $n$, the matrix algebra $\mathrm{M}_n(R)$ is an Azumaya algebra over $R$. The Brauer class of $\mathrm{M}_n(R)$ is the identity element of $\mathrm{Br}(R)$, since $\mathrm{M}_n(R)$ is isomorphic to $\mathrm{End}(R^n)$.

*Remark* 11.2.8 Let $R$ be a commutative *local* ring, with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. In this case, a finitely generated $R$-module is projective if and only if it is free, and therefore the definition of an Azumaya algebra can be simplified: an Azumaya algebra $A$ over $R$ is an $R$-algebra that is finitely generated and free as an $R$-module and such that $A \otimes_R k$ is a central simple algebra over $k$. Since the endomorphism algebra of a free $R$-module is isomorphic to a matrix algebra, two Azumaya algebras $A, B$ over $R$ are equivalent if and only if there are positive integers $m, n$ such that the $R$-algebras

$$A \otimes_R \mathrm{M}_m(R) \cong \mathrm{M}_m(A) \quad \text{and} \quad B \otimes_R \mathrm{M}_n(R) \cong \mathrm{M}_n(B)$$

are isomorphic.

In their definition of the Brauer group of a commutative ring $R$, Auslander and Goldman (1960, p. 381) use central separable algebras. For a proof that these are the same as our Azumaya algebras see Bass (1967, Theorem 3.4.1) or DeMeyer and Ingraham (1971, Theorem II.3.4).

## 11.3 Properties and examples

We first consider the effect of base change on the Brauer group. Let $f\colon R \to S$ be a homomorphism of commutative rings.

**Lemma 11.3.1** *Let $A$ be an Azumaya algebra over $R$. Then $A \otimes_R S$ is an Azumaya algebra over $S$.*

*Proof* We first show that $A \otimes_R S$ is a faithful $S$-module. If $\mathfrak{q}$ is any prime ideal of $S$, then $\mathfrak{p} = f^{-1}\mathfrak{q}$ is a prime ideal of $R$. The canonical isomorphisms

$$(A \otimes_R S)_{\mathfrak{q}} \cong (A \otimes_R S) \otimes_S S_{\mathfrak{q}} \cong A \otimes_R S_{\mathfrak{q}} \cong (A \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$$

(*cf.* Bourbaki, 1998, Section II.4.3, proof of Proposition 4) show that the $\mathfrak{q}$-rank of $A \otimes_R S$ is equal to the $\mathfrak{p}$-rank of $A$. By Lemma 11.1.2, it follows that $A \otimes_R S$ is faithful.

The homomorphism $\phi\colon A \otimes_R A^{\mathrm{opp}} \to \mathrm{End}_R A$ is surjective, and therefore its base change $\phi \otimes_R S$ is also surjective; the commutative diagram (11.1) then shows that $A \otimes_R S$ is an Azumaya algebra. □

It is straightforward to check that, if $A$ and $B$ are equivalent Azumaya algebras over $R$, then $A \otimes_R S$ and $B \otimes_R S$ are equivalent Azumaya algebras over $S$. We conclude that the ring homomorphism $f$ induces a homomorphism

$$f_*\colon \mathrm{Br}(R) \longrightarrow \mathrm{Br}(S)$$

between the Brauer groups of $R$ and $S$. If $g\colon S \to T$ is a second homomorphism of commutative rings, then the homomorphisms $g_* f_*$ and $(gf)_*$ coincide. Thus, Br defines a covariant functor from the category of commutative rings to the category of abelian groups.

We now generalize the notion of quaternion algebras over a field to what we call Hamilton algebras over a commutative ring. While not all Hamilton algebras are Azumaya algebras, many of the examples that we will consider are of this form.

**Definition 11.3.2** If $a, b$ are elements of a commutative ring $R$, then the *Hamilton algebra* $(a,b)_R$ is the $R$-algebra freely generated by 1, $i$, $j$, $ij$ as an $R$-module and with multiplication uniquely determined by

$$i^2 = a, \quad j^2 = b, \quad ji = -ij.$$

If $f\colon R \to S$ is a homomorphism of commutative rings and if $a, b$ are elements of $R$, then the natural homomorphism

$$(a,b)_R \otimes_R S \to \big(f(a), f(b)\big)_S$$

is an isomorphism of algebras over $S$.

If $k$ is a field of characteristic different from 2, and $a, b$ are non-zero elements of $k$, then the Hamilton algebra $(a,b)_k$ over $k$ is the quaternion algebra of Definition 10.2.4. We warn the reader that, in general, an Hamilton algebra need not be an Azumaya algebra, even in the case in which $a$ and $b$ are units in $R$, as some of the following examples show.

**Example 11.3.3** Let $A$ be the Hamilton algebra $(3,5)_{\mathbf{Z}}$ over $\mathbf{Z}$. The algebra $A \otimes \mathbf{Z}/3\mathbf{Z}$ is the Hamilton algebra $(0,-1)_{\mathbf{F}_3}$ over $\mathbf{F}_3$. This algebra is not simple, since the subspace spanned by $i$ and $ij$ is a two-sided ideal. It follows that $A$ is not an Azumaya algebra over $\mathbf{Z}$. Of course, as we saw in Section 10.3, the Hamilton algebra $A \otimes \mathbf{Q} = (3,5)_{\mathbf{Q}}$ *is* a central simple algebra over $\mathbf{Q}$.

**Example 11.3.4** Let $A$ be the Hamilton algebra $(-1,-1)_{\mathbf{Z}}$ over $\mathbf{Z}$. The algebra $A \otimes \mathbf{Z}/2\mathbf{Z}$ is the Hamilton algebra $(1,1)_{\mathbf{F}_2}$ over $\mathbf{F}_2$. This algebra is not central, since it is commutative (as is any Hamilton algebra over a ring of characteristic 2). Therefore the Hamilton algebra $A$ is not an Azumaya algebra over $\mathbf{Z}$.

**Example 11.3.5** Let $R$ be the ring $\mathbf{Q}[x, x^{-1}]$ and $A$ the Hamilton algebra $(-1,x)_R$. For every maximal ideal $\mathfrak{m}$ of $R$, the image of $x$ in the residue field $R/\mathfrak{m}$ is non-zero. Since the residue field has characteristic different from 2, the algebra $A \otimes R/\mathfrak{m}$ is a quaternion algebra and hence a central simple algebra over $R/\mathfrak{m}$. By Proposition 11.2.3 we deduce that $A$ is an Azumaya algebra over $R$. We claim that the class of the Azumaya algebra $A$ is not in the image of homomorphism $\iota_*\colon \mathrm{Br}(\mathbf{Q}) \to \mathrm{Br}(R)$ induced by the inclusion $\iota\colon \mathbf{Q} \to R$. Indeed, for any maximal ideal $\mathfrak{m}$ of $R$, the composition

$$\mathrm{Br}(\mathbf{Q}) \xrightarrow{\iota_*} \mathrm{Br}(R) \longrightarrow \mathrm{Br}(R/\mathfrak{m})$$

is induced by the inclusion $\mathbf{Q} \to R/\mathfrak{m}$. Therefore, if $R/\mathfrak{m}$ is isomorphic to $\mathbf{Q}$, then this composition is the identity. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the ideals generated by $x-1$ and $x+1$, respectively. Since the image of $A$ in $\mathrm{Br}(R/\mathfrak{m}) \cong \mathrm{Br}(\mathbf{Q})$ is the trivial class of $(-1,1)_{\mathbf{Q}}$, while its image in $\mathrm{Br}(R/\mathfrak{n}) \cong \mathrm{Br}(\mathbf{Q})$ is the non-trivial class of $(-1,-1)_{\mathbf{Q}}$, we conclude that the class of $A$ in $\mathrm{Br}(R)$ is not in the image of $\iota_*$. In the language of Chapter 12, we have just exhibited a non-constant class in $\mathrm{Br}(\mathbf{A}^1_{\mathbf{Q}} \smallsetminus \{0\})$.

Our next result is a "spreading out" result for Azumaya algebras.

**Proposition 11.3.6** *Let $R$ be a commutative ring and let $A$ be an $R$-algebra that is finitely generated and projective as an $R$-module. Suppose that $\mathfrak{p}$ is a prime ideal of $R$ such that $A \otimes k(\mathfrak{p})$ is a central simple algebra over $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Then there is an element $f$ in $R \smallsetminus \mathfrak{p}$ such that $A \otimes R[f^{-1}]$ is an Azumaya algebra over $R[f^{-1}]$.*

*Proof* Firstly, the corollary to Proposition 2 of Bourbaki (1998, Section II.5.1) shows that there exists $s \in R \smallsetminus \mathfrak{p}$ such that $A \otimes R[s^{-1}]$ is *free* over $R[s^{-1}]$. Using the isomorphism $R[s^{-1}][t^{-1}] \cong R[(st)^{-1}]$ and replacing $R$ by $R[s^{-1}]$, we reduce to the case where $A$ is free over $R$.

So now let $A$ be an $R$-algebra that is finitely generated and free as an $R$-module, and suppose that $A(\mathfrak{p}) := A \otimes_R k(\mathfrak{p})$ is a central simple algebra over $k(\mathfrak{p})$. (In particular, this means that $A$ has non-zero rank.) Denote by $q$ the composite homomorphism $R \to R_{\mathfrak{p}} \to k(\mathfrak{p})$. Pick a basis for $A$ as an $R$-module. This induces bases for $A \otimes_R A^{\mathrm{opp}}$ and $\mathrm{End}_R A$; let $\mathbf{M}$ be the $n^2 \times n^2$ matrix, with entries in $R$, representing the homomorphism $\phi$. By Proposition 10.2.14, the homomorphism $A(\mathfrak{p}) \otimes_{k(\mathfrak{p})} A(\mathfrak{p})^{\mathrm{opp}} \to \mathrm{End}_{k(\mathfrak{p})} A(\mathfrak{p})$ is an isomorphism; by diagram (11.1), this homomorphism is just the base change of $\phi$ to $k(\mathfrak{p})$ and so is represented by a matrix obtained by simply applying $q$ to all the entries of $\mathbf{M}$. It follows that $q(\det \mathbf{M})$ is non-zero and therefore that $\det \mathbf{M}$ does not lie in $\mathfrak{p}$. Set $f = \det \mathbf{M}$ and let $S$ be $R[f^{-1}]$; then $\mathbf{M}$ has an inverse over the ring $S$, and so (again by diagram (11.1)) the map

$$A_S \otimes_S (A_S)^{\mathrm{opp}} \to \mathrm{End}_S A_S$$

is an isomorphism. Also, $A_S$ is free of non-zero rank and hence faithful over $S$, meaning that $A_S$ is an Azumaya algebra over $S$. □

**Example 11.3.7** In Example 11.3.4 we saw that the Hamilton algebra $(-1,-1)_{\mathbf{Z}}$ is not an Azumaya algebra over $\mathbf{Z}$. However, it is true that $(-1,-1)_{\mathbf{Z}} \otimes \mathbf{Z}/p\mathbf{Z}$ is a central simple algebra over $\mathbf{F}_p$ for any odd prime $p$, and also that $(-1,-1)_{\mathbf{Z}} \otimes \mathbf{Q}$ is a central simple algebra over $\mathbf{Q}$. It follows that Proposition 11.3.6 applies with $\mathfrak{p} = (p)$ for $p$ odd, and also for $\mathfrak{p} = (0)$. In fact $(-1,-1)_{\mathbf{Z}} \otimes \mathbf{Z}[\frac{1}{2}]$ is an Azumaya algebra over $\mathbf{Z}[\frac{1}{2}]$.

**Exercise 11.3.8** Taking $A = (-1,-1)_{\mathbf{Z}}$ as in the previous example, write down the $4 \times 4$ integer matrix representing the $\mathbf{Z}$-linear transformation $x \mapsto axa'$ on $A$ for $a, a'$ each taking the values $1, i, j, ij$. Put the entries of these 16 matrices as columns in a $16 \times 16$ matrix, which represents the linear transformation

$$A \otimes_{\mathbf{Z}} A^{\mathrm{opp}} \to \operatorname{End} A, \quad (a \otimes a') \mapsto (x \mapsto axa').$$

Verify that this $16 \times 16$ matrix has determinant $2^{16}$.

**Corollary 11.3.9** *Let $R$ be an integral domain and let $\mathfrak{p}$ be a prime ideal in $R$. Let $A'$ be an Azumaya algebra over the local ring $R_{\mathfrak{p}}$. Then there exist $f \in R \smallsetminus \mathfrak{p}$ and an Azumaya algebra $A$ over $R[f^{-1}]$ satisfying $A \otimes_{R[f^{-1}]} R_{\mathfrak{p}} \cong A'$.*

*Proof* Because $R_{\mathfrak{p}}$ is local, $A'$ is a free module over $R_{\mathfrak{p}}$. Let $x_1, \ldots, x_n$ be a basis for $A'$, and define the corresponding *structure constants* $c^{\gamma}_{\alpha\beta} \in R_{\mathfrak{p}}$ by the formula $x_\alpha x_\beta = \sum_\gamma c^{\gamma}_{\alpha\beta} x_\gamma$. Take $g \in R \smallsetminus \mathfrak{p}$ to be such that $gc^{\gamma}_{\alpha\beta} \in R$ for all $i, j, k$; then all the $c^{\gamma}_{\alpha\beta}$ lie in the subring $R[g^{-1}] \subset R_{\mathfrak{p}}$. It follows that the $R[g^{-1}]$-submodule $B \subset A'$ generated by $x_1, \ldots, x_n$ is closed under multiplication, so is a $R[g^{-1}]$-subalgebra of $A'$. Since $R[g^{-1}]$ injects into $R_{\mathfrak{p}}$, we see that $x_1, \ldots, x_n$ are linearly independent over $R[g^{-1}]$ and therefore $B$ is free. By construction, $B$ satisfies the hypotheses of Proposition 11.3.6. We obtain $h \in R[g^{-1}] \smallsetminus \mathfrak{p}[g^{-1}]$ such that $A = B \otimes_R R[g^{-1}][h^{-1}]$ is an Azumaya algebra over $R[g^{-1}][h^{-1}]$. Writing $h = h'g^{-r}$ with $h \in R$ and setting $f = gh'$, we have $R[g^{-1}][h^{-1}] = R[f^{-1}]$, and $A$ is an $R[f^{-1}]$-algebra with the desired properties. □

The special case of Corollary 11.3.9 in which $\mathfrak{p}$ is the zero ideal is worth stating in its own right.

**Corollary 11.3.10** *Let $R$ be an integral domain with field of fractions $K$. Suppose that $A'$ is a central simple algebra over $K$. Then there exist non-zero $f \in R$ and an Azumaya algebra $A$ over $R[f^{-1}]$ satisfying $A \otimes K \cong A'$.*

Next, we examine Azumaya algebras over a complete local ring, beginning with an example.

**Example 11.3.11** In Example 10.2.6 we showed that the quaternion algebra $A = (-1,3)_{\mathbf{F}_7}$ is isomorphic to a matrix algebra. Using Hensel's Lemma (Theorem 2.1.3) we now show that the Hamilton algebra $A' = (-1,3)_{\mathbf{Z}_7}$ is isomorphic to the matrix algebra $\mathrm{M}_2(\mathbf{Z}_7)$. Firstly, note that $A' \otimes_{\mathbf{Z}_7} \mathbf{F}_7$ and $A$ are isomorphic (and in particular $A'$ is an Azumaya algebra). Next, we find matrices $I', J' \in \mathrm{M}_2(\mathbf{Z}_7)$ satisfying $(I')^2 = -\mathrm{Id}$, $(J')^2 = 3\mathrm{Id}$ and $I'J' + J'I' = 0$, as follows. Exactly as before, we set

$$I' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and try to solve for $a, b, c, d$. Again, the equation $I'J' + J'I' = 0$ translates to $a + d = 0$, $b = c$, and the equation $(J')^2 = 3\mathrm{Id}$ then becomes equivalent to the equation $a^2 + b^2 = 3$. We know that $a = 1$ and $b = 3$ is a solution modulo 7, and we can therefore set $a = 1$ and use Hensel's Lemma to solve the equation $1 + b^2 = 3$ in $\mathbf{Z}_7$, thereby finding $J'$. Hence, the matrices $I', J'$ determine a homomorphism $\rho' \colon A' \to \mathrm{M}_2(\mathbf{Z}_7)$ of algebras over $\mathbf{Z}_7$, as in Example 10.2.6. To show that $\rho'$ is an isomorphism, observe that it is a homomorphism of free $\mathbf{Z}_7$-modules that is an isomorphism modulo 7. Thus, $\rho'$ is a $\mathbf{Z}_7$-linear map with invertible determinant, and therefore is an isomorphism.

The example we just discussed is a manifestation of a general fact: that the Brauer group of a complete local ring injects into the Brauer group of its residue field, which can be proved by an extension of the argument we just gave. The following proposition states that the map of Brauer groups is actually an isomorphism.

**Proposition 11.3.12** *Let $R$ be a complete, commutative local ring, with maximal ideal $\mathfrak{m}$. The natural map $\mathrm{Br}\, R \to \mathrm{Br}(R/\mathfrak{m})$ is an isomorphism.*

*Proof* See Auslander and Goldman (1960, Theorem 6.5). □

**Corollary 11.3.13** *Let $p$ be a prime, let $k$ be a finite extension of $\mathbf{Q}_p$, and let $R$ be the ring of integers in $k$. Then $\mathrm{Br}\, R$ is trivial.*

*Proof* The residue field of $R$ is finite, and thus has trivial Brauer group by Corollary 10.4.10. □

**Exercise 11.3.14** Show that the Brauer group of a finite ring is trivial. (Hint: Reduce to the local case and observe that a finite local ring is complete.)

We now state an important theorem, due to Auslander and Goldman, comparing the Brauer group of a regular integral domain to the Brauer group of its field of fractions. This result will be essential to our definition of the Brauer group of a variety in the next chapter.

**Theorem 11.3.15** *If $R$ is a regular integral domain with field of fractions $K$, then the natural homomorphism $\mathrm{Br}\, R \to \mathrm{Br}\, K$ is injective.*

*Proof* See Auslander and Goldman (1960, Theorem 7.2). □

When $R$ is a regular integral domain, we will often think of the Brauer group of $R$ as being a subgroup of the Brauer group of its fraction field.

**Example 11.3.16** Auslander and Goldman also give an example showing that the assumption of regularity cannot be removed from Theorem 11.3.15. Indeed, set $R = \mathbf{R}[x,y]/(x^2+y^2)$ and let $A$ be the Hamilton algebra $(-1,-1)_R$. Since $\mathbb{H}_{\mathbf{R}} = (-1,-1)_{\mathbf{R}}$ is an Azumaya algebra over $\mathbf{R}$ and $A$ is isomorphic to $\mathbb{H}_{\mathbf{R}} \otimes R$, it follows that $A$ is an Azumaya algebra over $R$. Denote by $\mathfrak{m}$ the maximal ideal $(x,y)$ of $R$. The algebra $A \otimes R/\mathfrak{m}$ is isomorphic to $\mathbb{H}_{\mathbf{R}}$ and therefore represents the non-trivial element of $\mathrm{Br}(R/\mathfrak{m})$. We deduce that $A$ represents a non-trivial element of $\mathrm{Br}\,R$. On the other hand, in the fraction field $K$ of $R$ the element $x/y$ is a square root of $-1$ and therefore $A \otimes K$ is isomorphic to $\mathrm{M}_2(K)$. We conclude that the homomorphism $\mathrm{Br}\,R \to \mathrm{Br}\,K$ is not injective.

We finish with an algebraic statement which, while not essential to our treatment of the Brauer group, is closely related to the ideas of the following section. First, we prove a patching result for algebras and an extension result for projective modules.

**Lemma 11.3.17** (Patching) *Let $R$ be a ring, and let $f_1, f_2 \in R$ be two elements satisfying $R = Rf_1 + Rf_2$. Let $A_1$,$A_2$ be Azumaya algebras over $R_{f_1}$ and $R_{f_2}$ respectively, and suppose that $\phi\colon (A_2)_{f_1} \to (A_1)_{f_2}$ is an isomorphism of $R_{f_1f_2}$-algebras. Then there exists an Azumaya algebra $A$ over $R$ such that $A_{f_1}$ is isomorphic to $A_1$, and $A_{f_2}$ is isomorphic to $A_2$.*

*Proof* Consider both $A_1$ and $A_2$ as $R$-modules. The map $A_1 \times A_2 \to (A_1)_{f_2}$ given by $\psi(a_1,a_2) = a_1 - \phi(a_2)$ is an $R$-module homomorphism; define $A$ to be its kernel. Thus $A$ is the subset of $A_1 \times A_2$ consisting of pairs $(a_1,a_2)$ satisfying $a_1 = \phi(a_2)$. Define multiplication on $A_1 \times A_2$ componentwise; the fact that $\phi$ is an isomorphism of algebras shows that $A$ is closed under multiplication and so is an $R$-algebra. It is straightforward to check that $A_{f_1}$ is isomorphic to $A_1$, and therefore $A_\mathfrak{p}$ is isomorphic to $(A_1)_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of $R$ not containing $f_1$; similarly, $A_\mathfrak{p}$ is isomorphic to $(A_2)_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ not containing $f_2$. By hypothesis, no prime ideal contains both $f_1$ and $f_2$. Since $A_1$ and $A_2$ are both Azumaya algebras, we deduce by Proposition 11.2.3 that $A$ is an Azumaya algebra over $R$. □

Notice that, in the special case where $A_1$ and $A_2$ are submodules of the same algebra over $R_{fg}$ and $\phi$ is the identity, we are just defining $A$ to be the intersection of $A_1$ and $A_2$.

**Exercise 11.3.18** Prove the following general patching statement. Let $f_1,\ldots,f_n$

be elements of $R$ that generate $R$ as an ideal, let $A_1, \dots, A_n$ be Azumaya algebras over $R_{f_1}, \dots, R_{f_n}$ respectively, and suppose that there are isomorphisms $\phi_{ij} \colon (A_i)_{f_j} \to (A_j)_{f_i}$ for all $i, j$ satisfying $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ (over $R_{f_i f_j f_k}$) for all $i, j, k$. Then there is an Azumaya algebra $A$ over $R$ such that $A_{f_i}$ is isomorphic to $A_i$ for all $i$.

**Lemma 11.3.19** *Let $R$ be a regular integral Noetherian domain of Krull dimension $d$. Let $S$ be a multiplicative subset of $R$ and let $M$ be a finitely generated projective $S^{-1}R$-module. If the rank of $M$ is greater than $d$, then there is a finitely generated projective $R$-module $\tilde{M}$ such that $S^{-1}\tilde{M}$ and $M$ are isomorphic.*

*Proof* Let $x_1, \dots, x_n$ be generators for $M$ as an $S^{-1}R$-module and let $N$ be the $R$-submodule of $M$ spanned by $x_1, \dots, x_n$. By construction, $N$ is finitely generated and $S^{-1}N$ is isomorphic to $M$; however, $N$ is not necessarily projective. Since $R$ is regular of finite Krull dimension, it follows by Auslander and Buchsbaum (1957, Corollary 4.8) that $N$ admits a finite resolution

$$0 \to P_r \to \cdots \to P_0 \to N \to 0$$

by projective $R$-modules $P_0, \dots, P_r$. Moreover, since $R$ is Noetherian, we may assume that $P_0, \dots, P_r$ are finitely generated: see Bass (1968, after Proposition 6.3). Localising the previous sequence at $S$ we obtain a projective resolution of $M$ which is therefore split and we deduce the isomorphism

$$M \oplus \bigoplus_{i \text{ odd}} S^{-1}P_i \cong \bigoplus_{i \text{ even}} S^{-1}P_i. \tag{11.2}$$

Since the $R$-modules $P_0, \dots, P_r$ are projective and finitely generated, if follows that there is a finitely generated projective $R$-module $Q$ such that $Q \oplus \bigoplus_{i \text{ odd}} P_i$ is free; let $q$ be its rank. Adding $S^{-1}Q$ to both sides of the isomorphism in (11.2) we find

$$M \oplus (S^{-1}R)^q \cong S^{-1}P',$$

where $P' \cong Q \oplus \bigoplus_{i \text{ even}} P_i$ is finitely generated and projective of rank greater than $d + q$. Using item (a) in Weibel (2013, Bass–Serre Cancellation Theorem I.2.3) we deduce that there is a projective $R$-module $\tilde{M}$ and an isomorphism $P' \cong \tilde{M} \oplus R^q$. Since the dimension of $S^{-1}R$ is at most the dimension of $R$, we can apply item (b) in the Bass–Serre Theorem to conclude that the $S^{-1}R$-modules $M$ and $S^{-1}\tilde{M}$ are isomorphic and the proof is complete. □

**Proposition 11.3.20** *Let $R$ be a regular integral Noetherian domain of finite*

*Krull dimension, with field of fractions $K$. Then there is an equality*

$$\operatorname{Br} R = \bigcap_{\mathfrak{m}} \operatorname{Br} R_{\mathfrak{m}}$$

*where the intersection is over all maximal ideals* $\mathfrak{m}$ *in $R$, and all Brauer groups are considered as subgroups of* $\operatorname{Br} K$.

*Proof* We give a proof following Hoobler (1980).

Let $\alpha$ be a class in $\operatorname{Br} K$ lying in $\operatorname{Br} R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of $R$; we must show that $\alpha$ lies in $\operatorname{Br} R$. Define a set $S = \{f \in R \mid \alpha \in \operatorname{Br} R_f\}$. Corollary 11.3.9 shows that, for any maximal ideal $\mathfrak{m}$, there is an element $f$ of $R \setminus \mathfrak{m}$ contained in $S$; thus $S$ is not contained in any maximal ideal of $R$. The plan is to show that $S$ is an ideal; then it will follow that $S$ contains $1_R$, and therefore that $\alpha$ lies in $\operatorname{Br} R$.

We first show that $S$ is closed under multiplication by elements of $R$. If $f$ is in $S$, and $g$ is any element of $R$, then the inclusion $R_f \to K$ factors through $R_{fg}$, and so we have inclusions $\operatorname{Br} R_f \subseteq \operatorname{Br} R_{fg} \subset \operatorname{Br} K$ and $fg$ lies in $S$.

Now let $f$ and $g$ be two elements of $S$; we must show that $f+g$ also lies in $S$. We reduce to the case in which $f+g$ is invertible by replacing $R$ by $R_{f+g}$. By the definition of $S$, there are Azumaya algebras $A_f$ over $R_f$ and $A_g$ over $R_g$, such that $A_f \otimes K$ and $A_g \otimes K$ both have class $\alpha$ in $\operatorname{Br} K$. Because $\operatorname{Br} R_{fg} \to \operatorname{Br} K$ is injective, it follows that $(A_f)_g$ and $(A_g)_f$ have the same class in $\operatorname{Br} R_{fg}$. Therefore there are finitely generated, projective modules $P$, $Q$ over $R_{fg}$, of non-zero rank, and an isomorphism $(A_f)_g \otimes_{R_{fg}} \operatorname{End} P \cong (A_g)_f \otimes_{R_{fg}} \operatorname{End} Q$. Let $V$ be a free $R_{fg}$-module of sufficiently large rank. From Lemma 11.3.19 we obtain a finitely generated, projective $R_f$-module $\tilde{P}$ and a finitely generated, projective $R_g$-module $\tilde{Q}$ such that $\tilde{P} \otimes_{R_f} R_{fg}$ and $P \otimes_{R_{fg}} V$ are isomorphic, and $\tilde{Q} \otimes_{R_g} R_{fg}$ and $Q \otimes_{R_{fg}} V$ are isomorphic. Thus we have an isomorphism

$$(A_f \otimes_{R_f} \operatorname{End} \tilde{P})_g \cong (A_g \otimes_{R_g} \operatorname{End} \tilde{Q})_f$$

of $R_{fg}$-algebras. To conclude, we apply Lemma 11.3.17 to the algebras $A_f \otimes_{R_f} \operatorname{End} \tilde{P}$ and $A_g \otimes_{R_g} \operatorname{End} \tilde{Q}$ obtaining an Azumaya algebra $A$ over $R$ whose class in $\operatorname{Br} K$ is $\alpha$. ☐

# 12

# The Brauer group of a variety

In this chapter we define and study the Brauer group of a smooth, geometrically irreducible variety. Following the motivation of Section 10.5, the Brauer group of a variety $X$ will be the subgroup of $\operatorname{Br}\kappa(X)$ consisting of those elements that can be evaluated at all points of $X$.

Defining the Brauer group is straightforward, but computing the Brauer groups of specific varieties is a very different matter. Even the question of whether the Brauer group is finite is open for many classes of varieties. In this chapter we will give some results describing the Brauer groups of curves, affine and projective spaces. Later, in Chapter 15, we will see how to use Galois cohomology to compute the Brauer groups of other varieties, such as rational varieties.

## 12.1 Definition of the Brauer group

Let $k$ be a field and $\bar{k}$ an algebraic closure of $k$. Recall that for a variety $X$ over $k$ and a point $x$ in $X(\bar{k})$, we denote by $\mathscr{O}_{X,x}$ the local ring of $X$ at $x$. If $X$ is irreducible and $x$ is a smooth point, then the ring $\mathscr{O}_{X,x}$ is a regular integral domain, so that by Theorem 11.3.15 the group $\operatorname{Br}\mathscr{O}_{X,x}$ can be viewed as a subgroup of $\operatorname{Br}\kappa(X)$. Elements of $\operatorname{Br}\kappa(X)$ lying in the image of $\operatorname{Br}\mathscr{O}_{X,x}$ are said to be *unramified* at $x$, and elements of $\operatorname{Br}\kappa(X)$ lying outside the image of $\operatorname{Br}\mathscr{O}_{X,x}$ are said to be *ramified* at $x$.

**Definition 12.1.1** Let $X$ be a smooth irreducible variety over $k$. The *Brauer group* $\operatorname{Br}X$ of $X$ is the subgroup of $\operatorname{Br}\kappa(X)$ consisting of those elements that are unramified at all points of $X$, that is,

$$\operatorname{Br}X = \bigcap_{x\in X(\bar{k})} \operatorname{Br}\mathscr{O}_{X,x},$$

where the intersection is taken in the group $\operatorname{Br}\kappa(X)$.

**Example 12.1.2** Let $X$ be the non-singular del Pezzo surface of degree 4 of Example 2.3.5 defined by the equations

$$\begin{cases} uv = x^2 - 5y^2 \\ (u+v)(u+2v) = x^2 - 5z^2 \end{cases}$$

and let $\mathscr{A}$ be the quaternion algebra

$$\mathscr{A} = \left(5, \frac{u}{u+v}\right)$$

over $\kappa(X)$. We claim that the class of $\mathscr{A}$ in $\operatorname{Br}\kappa(X)$ lies in $\operatorname{Br}X$. To see this, first let $P$ be a point of $X$ where neither $u$ nor $u+v$ vanishes. Then the Hamilton algebra over the local ring $\mathscr{O}_{X,P}$ defined by

$$\mathscr{A}_P = \left(5, \frac{u}{u+v}\right)_{\mathscr{O}_{X,P}}$$

is an Azumaya algebra, because its reduction modulo the maximal ideal of $\mathscr{O}_{X,P}$ is the central simple algebra $(5, u(P)/(u(P)+v(P)))_{k_P}$, where $k_P$ is the residue field at $P$. Moreover, we have $\mathscr{A}_P \otimes_{\mathscr{O}_{X,P}} \kappa(X) = \mathscr{A}$, and so the class of $\mathscr{A}$ in $\operatorname{Br}\kappa(X)$ lies in the image of $\operatorname{Br}\mathscr{O}_{X,P}$.

To repeat the argument for points where either $u$ or $u+v$ vanishes, we do some arithmetic in the function field $\kappa(X)$. Dividing both sides of the equation $uv = x^2 - 5y^2$ by $v^2$, we see that

$$\frac{u}{v} = \frac{x^2 - 5y^2}{v^2} = N_{\kappa(X)(\sqrt{5})/\kappa(X)}\left(\frac{x + \sqrt{5}y}{v}\right)$$

is a norm from $\kappa(X)(\sqrt{5})$, and so by Exercise 10.2.5(iv) the quaternion algebra $(5, v/(u+v))_{\kappa(X)}$ is isomorphic to $\mathscr{A}$. In a similar way, we get the following four quaternion algebras over $\kappa(X)$, all isomorphic:

$$\mathscr{A} = \left(5, \frac{u}{u+v}\right), \quad \left(5, \frac{v}{u+v}\right), \quad \left(5, \frac{u}{u+2v}\right), \quad \left(5, \frac{v}{u+2v}\right). \qquad (12.1)$$

Repeating the argument above with these isomorphic algebras shows that the class of $\mathscr{A}$ in $\operatorname{Br}\kappa(X)$ lies in the image of $\operatorname{Br}\mathscr{O}_{X,P}$ for all points $P$ of $X$, except possibly when $u(P)$ and $v(P)$ both vanish. That leaves only the four points $[0,0,\sqrt{5},\pm 1,\pm 1]$ where the class of $\mathscr{A}$ might be ramified.

To show that $\mathscr{A}$ is indeed unramified at these final four points, we produce yet another quaternion algebra isomorphic to $\mathscr{A}$. Let $f \in \kappa(X)$ and

$g \in \kappa(X)(\sqrt{5})$ be the functions

$$f = \frac{u}{u+v}, \qquad g = \frac{u+v+x-\sqrt{5}z}{u+x-\sqrt{5}y}.$$

Then we have

$$\begin{aligned} fN(g) &= \frac{u((u+v+x)^2-5z^2)}{(u+v)((u+x)^2-5y^2)} \\ &= \frac{u((u+v)^2+2x(u+v)+x^2-5z^2)}{(u+v)(u^2+2ux+x^2-5y^2)} \\ &= \frac{u((u+v)^2+2x(u+v)+(u+v)(u+2v))}{(u+v)(u^2+2ux+uv)} \\ &= \frac{2u+3v+2x}{u+v+2x}. \end{aligned}$$

Therefore $\mathscr{A}$ is isomorphic to the quaternion algebra

$$\left(5, \frac{2u+3v+2x}{u+v+2x}\right)_{\kappa(X)}$$

which by the argument above is unramified at the remaining four points of $X$.

The procedure followed in the example above may seem rather *ad hoc*. In Proposition 15.3.3, we shall see a general criterion for deciding when a cyclic algebra over the function field of a variety lies in the Brauer group of the variety. The calculation above may be viewed as an explicit form of the proof of that proposition.

*Remark* 12.1.3 To show that the class of an algebra $\mathscr{A} \in \operatorname{Br}\kappa(X)$ lies in $\operatorname{Br}X$, we need to find, for each point $x \in X$, an Azumaya algebra $A_x$ over $\mathscr{O}_{X,x}$ such that $A_x \otimes_{\mathscr{O}_{X,x}} \kappa(X)$ is Brauer-equivalent to $\mathscr{A}$. In Example 12.1.2, we were able to do rather more than this: starting with a central simple algebra $\mathscr{A}$ over $\kappa(X)$, we found, for each $x \in X$, an $\mathscr{O}_{X,x}$-subalgebra $A_x$ of $\mathscr{A}$ such that $A_x \otimes_{O_{X,x}} \kappa(X)$ was isomorphic to $\mathscr{A}$. In Auslander and Goldman (1960, proof of Proposition 7.4), the authors showed that this is always possible when $X$ is smooth and of dimension at most two; the proof relies on the fact that, over a two-dimensional regular local ring, a maximal order in a central simple algebra is always projective. Auslander and Goldman comment that "It is not known at the present time whether the restriction on the dimension of $R$ is actually necessary." More recently, Antieau and Williams (2014) have settled this question by showing that there is a smooth affine variety $X$ over the complex numbers of dimension 6 and a division algebra in $\operatorname{Br}X$ admitting no maximal orders which are Azumaya algebras.

*Remark* 12.1.4 If $X$ is a smooth irreducible *affine* variety over $k$ with coordinate ring $R$, then the groups $\operatorname{Br} X$ and $\operatorname{Br} R$ coincide. Indeed, it is clear from the definition that the image of $\operatorname{Br} R$ in $\operatorname{Br} \kappa(X)$ lies in $\operatorname{Br} X$. To see that this inclusion is an equality, it suffices to apply Proposition 11.3.20. This is one reason that we assumed $X$ to be smooth in Definition 12.1.1. In general, if $X$ is not smooth, then the natural maps $\operatorname{Br} \mathscr{O}_{X,x} \to \operatorname{Br} \kappa(X)$ need not be injective (as seen in Example 11.3.16).

Let $X$ be a smooth irreducible variety. If $U$ is a non-empty open subvariety of $X$, then the function fields $\kappa(U)$ and $\kappa(X)$ can be identified, and so there is a natural inclusion $\operatorname{Br} X \subseteq \operatorname{Br} U$. If $\mathscr{U}$ is an open cover of $X$, then the equality

$$\operatorname{Br} X = \bigcap_{U \in \mathscr{U}} \operatorname{Br} U$$

follows immediately from Definition 12.1.1. If each $U \in \mathscr{U}$ is affine with coordinate ring $R_U$, then by Remark 12.1.4 we have

$$\operatorname{Br} X = \bigcap_{U \in \mathscr{U}} \operatorname{Br} R_U,$$

where, as before, the intersection takes place in $\operatorname{Br} \kappa(X)$.

We now define what it means to evaluate a Brauer class in $\operatorname{Br} X$ at a point of $X$. Let $\ell$ be a field containing $k$ and let $x$ be a point in $X(\ell)$. Then $\operatorname{Br} X$ is contained in the image of the natural injection $\operatorname{Br} \mathscr{O}_{X,x} \to \operatorname{Br} \kappa(X)$: if $\ell$ is a subfield of $\bar{k}$ then this follows immediately from the definition of $\operatorname{Br} X$, and otherwise by **??**. Evaluation at the point $x$ defines a ring homomorphism $\mathscr{O}_{X,x} \to \ell$, inducing a group homomorphism $\operatorname{ev}_x \colon \operatorname{Br} \mathscr{O}_{X,x} \to \operatorname{Br} \ell$, which in turn restricts to a homomorphism

$$\operatorname{ev}_x \colon \operatorname{Br} X \to \operatorname{Br} \ell.$$

If $A$ is a class in $\operatorname{Br} X$ we sometimes denote $\operatorname{ev}_x(A)$ by $A(x)$. By Remark **??**, the evaluation map $\operatorname{ev}_x$ factors through $\operatorname{Br} \mathscr{O}_{X_\ell,x}$.

**Example 12.1.5** In Example 12.1.2, we looked at the quaternion algebra $\mathscr{A} = (5, u/(u+v))$ on the variety $X$ of Example 2.3.5. At any point $P$ of $X(\ell)$ where neither $u$ nor $u+v$ vanish, we saw that the algebra $(5, u/u+v)$ over $\mathscr{O}_{X,P}$ is an Azumaya algebra; evaluating at $P$ gives the central simple algebra $(5, u(P)/(u(P)+v(P)))_\ell$.

The example above illustrates a general principle: given an explicit description of a central simple algebra $\mathscr{A}$ over $\kappa(X)$, and a point $x$ of $X$, we can try to find $\mathscr{A}(x)$ by evaluating everything in sight at $x$; if we obtain a central simple algebra, then it is $\mathscr{A}(x)$. Let formulate this principle into a precise lemma. Notice first that, for any finitely generated free module $A$ with basis $\{e_1,\dots,e_n\}$ over a ring $R$, putting the structure of an $R$-algebra on $A$ is the same as specifying the $n^3$ *structure constants* $c^{\gamma}_{\alpha\beta} \in R$ defined by $e_\alpha e_\beta = \sum_\gamma c^{\gamma}_{\alpha\beta} e_\gamma$.

**Lemma 12.1.6** *Let $X$ be a smooth variety over a field $k$, and let $\mathscr{A}$ be a central simple algebra over $\kappa(X)$. Fix a basis $\{e_1,\dots,e_n\}$ for $\mathscr{A}$ over $\kappa(X)$, and let $c^{\gamma}_{\alpha\beta} \in \kappa(X)$ be the corresponding structure constants. Let $\ell$ be an extension of $k$, and let $x \in X(\ell)$ be a point. Suppose that each $c^{\gamma}_{\alpha\beta}$ lies in $\mathscr{O}_{X,x}$. Let the $\ell$-algebra $B$ be the vector space $\ell^n$ endowed with the multiplicative structure defined by the structure constants $c^{\gamma}_{\alpha\beta}(x)$. If $B$ is a central simple algebra, then the class of $\mathscr{A}$ in $\mathrm{Br}\,X$ is unramified at $x$, and $\mathscr{A}(x)$ is isomorphic to $B$.*

*Proof* Let $A_x$ be the $\mathscr{O}_{X,x}$-submodule of $\mathscr{A}$ spanned by $\{e_1,\dots,e_n\}$. Since the $e_i$ are linearly independent over $\kappa(X)$, they are also linearly independent over $\mathscr{O}_{X,x}$, so form a basis for $A_x$. The fact that all the $c^{\gamma}_{\alpha\beta}$ lie in $\mathscr{O}_{X,x}$ means that $A_x$ is closed under multiplication and therefore is an $\mathscr{O}_{X,x}$-algebra. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathscr{O}_{X,x}$; the evaluation map at $x$ defines an isomorphism $\mathscr{O}_{X,x}/\mathfrak{m}_x \to \ell$. It follows easily from the definition of tensor product that there is an isomorphism $A_x \otimes_{\mathscr{O}_{X,x}} \ell \cong B$. If $B$ is a central simple algebra, then Proposition 11.2.3 shows that $A_x$ is an Azumaya algebra over $\mathscr{O}_{X,x}$. By construction, $A_x \otimes_{\mathscr{O}_{X,x}} \kappa(X)$ is isomorphic to $\mathscr{A}$, showing that the class of $\mathscr{A}$ is unramified at $x$, and $\mathscr{A}(x)$ is isomorphic to $B$ by definition. □

**Example 12.1.7** Let $X$ be a smooth variety over a field of characteristic zero. Let $\mathscr{A}$ be the quaternion algebra $(f,g)$ over $\kappa(X)$, defined by two non-zero functions $f,g \in \kappa(X)$. Taking the standard basis $\{1,i,j,ij\}$ for $\mathscr{A}$ gives 64 structure constants which are all either 0, $\pm 1$, $\pm f$, $\pm g$ or $\pm fg$. For any point $x \in X(\ell)$ where both $f$ and $g$ are defined and non-zero, $\mathscr{A}$ is unramified at $x$ and $\mathscr{A}(x)$ is the quaternion algebra $(f(x),g(x))$ over $\ell$.

**Proposition 12.1.8** *Let $X$ be a smooth, geometrically irreducible variety over a field $k$, and let $K/k$ be a field extension. Let $f\colon \mathrm{Br}\,\kappa(X) \to \mathrm{Br}\,\kappa(X_K)$ be the natural map. If a class $\alpha \in \mathrm{Br}\,\kappa(X)$ lies in $\mathrm{Br}\,X$, then its image $f(\alpha)$ lies in $\mathrm{Br}\,X_K$; thus $f$ induces a homomorphism $\mathrm{Br}\,X \to \mathrm{Br}\,X_K$.*

*Proof* Given a point $x \in X_K(\bar{K})$ of $X_K$, let $Z \subseteq X_K$ be the closure of $x$ in the $k$-Zariski topology on $X_K$. Since $Z$ is non-empty, the Nullstellensatz shows that $Z$ contains a point $y \in X(\bar{k})$. Under the inclusion $\kappa(X) \to \kappa(X_K)$, the image of the local ring $\mathscr{O}_{X,y}$ is contained in $\mathscr{O}_{X_K,x}$; so we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Br}\mathscr{O}_{X,y} & \longrightarrow & \operatorname{Br}\mathscr{O}_{X_K,x} \\ \downarrow & & \downarrow \\ \operatorname{Br}\kappa(X) & \xrightarrow{f} & \operatorname{Br}\kappa(X_K) \end{array}$$

If now $\alpha$ is a class of $\operatorname{Br}X$, then $\alpha$ lies in the image of $\operatorname{Br}\mathscr{O}_{X,y}$ and therefore $f(\alpha)$ lies in the image of $\operatorname{Br}\mathscr{O}_{X_K,x}$. This is true for all points $x$ of $X_K$, so $\alpha$ lies in $\operatorname{Br}X_K$. □

We will now define some subgroups of the Brauer group of a variety which relate in different ways to the base field $k$.

**Definition 12.1.9** Let $X$ be a smooth, geometrically irreducible variety over a field $k$.

(i) The image of the natural map $\operatorname{Br}k \to \operatorname{Br}\kappa(X)$ lies in $\operatorname{Br}X$, and is denoted $\operatorname{Br}_0 X$. Classes in $\operatorname{Br}_0 X$ are called *constant* classes.
(ii) The kernel of the natural map $\operatorname{Br}X \to \operatorname{Br}X^{\mathrm{sep}}$ is denoted $\operatorname{Br}_1 X$. Classes in $\operatorname{Br}_1 X$ are called *algebraic* classes.
(iii) Classes in $\operatorname{Br}X$ which are not algebraic are called *transcendental* classes.

Of course, every element of $\operatorname{Br}\kappa(X)$ is split by some finite extension of $\kappa(X)$. An element is algebraic if it is split by an extension of $\kappa(X)$ induced by a separable extension of the base field $k$. Note that every constant class is algebraic, because every element of $\operatorname{Br}k$ is split by a finite separable extension of $k$. We have inclusions $\operatorname{Br}_0 X \subset \operatorname{Br}_1 X \subset \operatorname{Br}X$.

**Example 12.1.10** The class of the algebra of Example 12.1.2 is algebraic, because that algebra splits if the base field is extended to $\mathbf{Q}(\sqrt{5})$.

**Example 12.1.11** Let $p$ be a prime and let $k$ be the field $\mathbf{F}_p(t)$. We construct a class $\alpha$ in $\operatorname{Br}\mathbf{A}^1_k$ with the following properties:

- $\alpha$ is split by an inseparable extension of $k$;
- $\alpha$ is split by a separable extension of $\kappa(\mathbf{A}^1_k)$;
- $\alpha$ is not split by any separable extension of $k$.

Therefore, $\alpha$ does not lie in $\operatorname{Br}_1 \mathbf{A}^1_k$.

Following Auslander and Goldman (1960, proof of Theorem 7.5), we define an algebra $A$ over $\kappa(\mathbf{A}^1_k) = k(T)$ of dimension $p^2$ generated as a $k(T)$-algebra by two elements $i, j$ satisfying the relations

$$i^p - i = T, \qquad j^p = t, \qquad ji = (i+1)j.$$

In other words, $A$ is the cyclic algebra $(k(T)(i)/k(T),\sigma,t)$ (see Definition 10.2.8), where $k(T)(i)$ is the Artin–Schreier extension defined by $i^p - i = T$ and $\sigma$ is the automorphism of $k(T)(i)/k(T)$ satisfying $\sigma(i) = i+1$. Then $A$ is split by the inseparable extension $k(T)(j)/k(T)$ induced by $k\big(t^{\frac{1}{p}}\big)/k$, and it is split by the separable extension $k(T)(i)/k(T)$. The class $\alpha$ of $A$ in $\mathrm{Br}\,k(T)$ is non-trivial, since an easy calculation shows that $t$ is not a norm for the extension $k(T)(i)/k(T)$. The class of the algebra $A$ lies in $\mathrm{Br}\,\mathbf{A}^1_k$ by Lemma 12.1.6, and it evaluates to the trivial class at the point $T = 0$, showing that the class of $A$ is not in $\mathrm{Br}_0\,\mathbf{A}^1_k$. However, Auslander and Goldman show that $\mathrm{Br}_1\,\mathbf{A}^1_k$ is equal to $\mathrm{Br}_0\,\mathbf{A}^1_k$ and hence the class of $A$ is not algebraic, that is, $A$ is not split by any separable extension of $k$.

**Exercise 12.1.12** Let $X$ be a smooth, geometrically irreducible variety over a field $k$.

(i) Suppose that $X$ has a $k$-point $x$. Show that the composition

$$\mathrm{Br}\,k \to \mathrm{Br}\,X \xrightarrow{\mathrm{ev}_x} \mathrm{Br}\,k$$

is the identity map. Deduce that, whenever $X(k)$ is non-empty, the natural map $\mathrm{Br}\,k \to \mathrm{Br}\,X$ is injective.

(ii) In the case in which $k$ is a number field, draw the same conclusion under the assumption that $X$ has points everywhere locally.

## 12.2 Properties of the Brauer group

We have said what it means for a Brauer class to be unramified at a point. Let us extend the definition and say what it means for a Brauer class to be unramified at any subvariety.

**Definition 12.2.1** Let $X$ be a smooth, geometrically irreducible variety over a field $k$, and let $Z$ be a subvariety of $X$. A class $\alpha \in \mathrm{Br}\,\kappa(X)$ is *unramified* at $Z$ if $\alpha$ lies in the image of the natural map $\mathrm{Br}\,\mathscr{O}_{X,Z} \to \mathrm{Br}\,\kappa(X)$.

*Remark* 12.2.2 (i) If $Z$ is a point of $X$, then this definition agrees with our previous one.

(ii) Because $X$ is smooth, the local ring $\mathscr{O}_{X,Z}$ is regular. By Theorem 11.3.15, the map $\mathrm{Br}\,\mathscr{O}_{X,Z} \to \mathrm{Br}\,\kappa(X)$ is injective.

(iii) Suppose that $Z \subset Y \subset X$ are subvarieties. Then we have an inclusion $\mathscr{O}_{X,Z} \subset \mathscr{O}_{X,Y}$ and therefore an inclusion $\mathrm{Br}\,\mathscr{O}_{X,Z} \subset \mathrm{Br}\,\mathscr{O}_{X,Y}$. In particular, if a class is unramified at $Z$ then it is unramified at $Y$. Since every subvariety contains a point of $X$, it follows that a class in $\mathrm{Br}\,X$ is unramified at all subvarieties.

We will now look at the behaviour of Brauer groups under morphisms of varieties.

**Proposition 12.2.3** *Let $f\colon X \to Y$ be a morphism of smooth, geometrically irreducible varieties over a field $k$. Then there is an induced morphism $f^*\colon \mathrm{Br}\,Y \to \mathrm{Br}\,X$, making* Br *into a contravariant functor from the category of such varieties to the category of Abelian groups.*

*Proof* We follow Colliot-Thélène (1995). Let $Z$ be the closure of the image of $f$, and let $\alpha$ be an element of $\mathrm{Br}\,Y$. Then $\alpha$ lies in $\mathrm{Br}\,\mathscr{O}_{Y,Z} \subset \mathrm{Br}\,\kappa(X)$. The residue field of the local ring $\mathscr{O}_{Y,Z}$ is the function field $\kappa(Z)$. Now $f\colon X \to Z$ is dominant, so induces a map of function fields $\kappa(Z) \to \kappa(X)$. We define $f^*(\alpha)$ to be the image of $\alpha$ under the homomorphism $\mathrm{Br}\,\mathscr{O}_{Y,Z} \to \mathrm{Br}\,\kappa(X)$ induced by the composite ring homomorphism $\mathscr{O}_{Y,Z} \to \kappa(Z) \to \kappa(X)$.

Let us show that $f^*(\alpha)$ lies in $\mathrm{Br}\,X$. Given any point $x$ of $X$, we have a commutative diagram of rings

$$\begin{array}{ccccc} \mathscr{O}_{Y,f(x)} & \longrightarrow & \mathscr{O}_{Z,f(x)} & \longrightarrow & \mathscr{O}_{X,x} \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{O}_{Y,Z} & \longrightarrow & \kappa(Z) & \longrightarrow & \kappa(X) \end{array}$$

where the vertical maps are inclusions, and the horizontal maps come from the morphisms $X \to Z \to Y$. There is therefore a corresponding diagram of Brauer groups, with the vertical maps still being inclusions. The class $\alpha$ lies in $\mathrm{Br}\,\mathscr{O}_{Y,Z}$; by assumption, $\alpha$ is unramified on $Y$ and so lies in $\mathrm{Br}\,\mathscr{O}_{Y,f(x)}$. By commutativity of the diagram, the class $f^*(\alpha) \in \mathrm{Br}\,\kappa(X)$ lies in $\mathrm{Br}\,\mathscr{O}_{X,x}$. Since this holds for any point $x$ of $X$, we deduce that $f^*(\alpha)$ lies in $\mathrm{Br}\,X$.

Verifying that this construction makes Br into a contravariant functor is an easy exercise. □

*Remark* 12.2.4 In the case that $f$ is the inclusion of an open subvariety $X$ into $Y$, this definition of $f^*$ is simply the inclusion $\mathrm{Br}\,Y \to \mathrm{Br}\,X$ given by regarding both as subgroups of $\mathrm{Br}\,\kappa(Y)$.

Let us now study the set of points at which a given Brauer class is unramified. The algebraic result of Corollary 11.3.9 can be interpreted geometrically as follows.

**Proposition 12.2.5** *Let $X$ be a smooth, geometrically irreducible variety over a field $k$, and let $Z$ be a subvariety of $X$. Let $\alpha$ be a class in* $\mathrm{Br}\,\kappa(X)$ *that is* *unramified at Z. Then there is a dense affine open subset U of X, having non-empty intersection with Z, such that* $\alpha$ *lies in* $\operatorname{Br} U$.

*Proof* Firstly, replacing $X$ by any dense affine subset meeting $Z$, we may assume that $X$ is affine. Let $R$ be the coordinate ring of $X$, so that $\kappa(X)$ is the field of fractions of $R$. Denote by $\mathfrak{p}$ the prime ideal of $R$ consisting of functions vanishing identically on $Z$; the local ring $\mathscr{O}_{X,Z}$ is then the localisation $R_{\mathfrak{p}}$.

By hypothesis, the class $\alpha$ lies in the image of $\operatorname{Br}\mathscr{O}_{X,Z}$ in $\operatorname{Br}\kappa(X)$. Let $A'$ be a central simple algebra over $\mathscr{O}_{X,Z}$ representing the class $\alpha$. By Corollary 11.3.9, there is a regular function $f \in R$, not vanishing identically on $Z$, and an Azumaya algebra $A$ over $R[f^{-1}]$, satisfying $A \otimes R_{\mathfrak{p}} \cong A'$. Thus the class $\alpha$ lies in the image of the map $\operatorname{Br} R[f^{-1}] \to \operatorname{Br}\kappa(X)$. Now let $U$ be the dense affine open subset of $X$ defined by $f \neq 0$; by construction, $U$ has non-empty intersection with $Z$. For any $x \in U$, we have inclusions $R[f^{-1}] \subset \mathscr{O}_{X,x} \subset \kappa(X)$; so the map $\operatorname{Br} R[f^{-1}] \to \operatorname{Br}\kappa(X)$ factors through $\operatorname{Br}\mathscr{O}_{X,x}$, showing that $\alpha$ lies in the image of $\operatorname{Br}\mathscr{O}_{X,x}$. Thus $\alpha$ is unramified at every point of $U$, that is, $\alpha$ lies in $\operatorname{Br} U$. □

**Corollary 12.2.6** *If* $\alpha$ *is any class in* $\operatorname{Br}\kappa(X)$*, then there is a dense open subset* $U \subset X$ *such that* $\alpha$ *lies in* $\operatorname{Br} U$.

*Proof* This is simply Proposition 12.2.5 in the case when $Z$ is the whole of $X$. □

**Corollary 12.2.7** *Let* $\alpha$ *be any class in* $\operatorname{Br}\kappa(X)$*. Then the set of points of X at which* $\alpha$ *is unramified is open in X (in the Zariski topology).*

*Proof* Let $V$ denote the set of points at which $\alpha$ is unramified. Proposition 12.2.5 shows that every point in $V$ has an open neighbourhood contained in $V$, and therefore $V$ is open. □

The closed subset of $X$ consisting of those points where an element $\alpha \in \operatorname{Br}\kappa(X)$ is ramified is called the *ramification locus* of $\alpha$. One of the deepest results in the theory of the Brauer group is the purity theorem, which we now state. Recall that a closed subset is said to be of pure codimension $c$ if each of its irreducible components is of codimension $c$.

**Theorem 12.2.8** (Purity Theorem) *Let X be a smooth, geometrically irreducible variety over a field k. Suppose either that X has dimension at most* 2*, or that k has characteristic zero.*

(i) *Let* $\alpha$ *be a class in* $\operatorname{Br}\kappa(X)$*. Then the ramification locus of* $\alpha$ *is either empty, or of pure codimension* 1 *in X.*

(ii) *If $Z$ is any closed subset of $X$ of codimension at least* 2*, then* $\mathrm{Br}(X \setminus Z)$ *coincides with* $\mathrm{Br}\, X$.

(iii) *There is an equality*

$$\mathrm{Br}\, X = \bigcap_{Z} \mathrm{Br}\, \mathscr{O}_{X,Z}$$

*where $Z$ runs over the prime divisors on $X$, and the intersection takes place in* $\mathrm{Br}\, \kappa(X)$.

*Proof* A proof of this theorem is far beyond the scope of this book, but let us show that the statements are equivalent. (More precisely, we will show that if one of the three statements holds for all smooth, geometrically irreducible varieties over a particular field, then so do the other two.)

Assume (i), and suppose that $\alpha$ lies in $\mathrm{Br}(X \setminus Z)$. Then the ramification locus of $\alpha$ is contained in $Z$, so cannot be of pure codimension 1; it is therefore empty. So $\alpha$ lies in $\mathrm{Br}\, X$, proving (i) $\Rightarrow$ (ii).

Now assume (ii), and suppose that $\alpha$ is a class lying in $\mathrm{Br}\, O_{X,Z}$ for all prime divisors $Z$. Then Proposition 12.2.5 shows that no $Z$ is contained in the ramification locus of $\alpha$. Therefore the ramification locus has codimension at least 2, so (ii) implies that $\alpha$ lies in $\mathrm{Br}\, X$, proving (ii) $\Rightarrow$ (iii).

Finally, assume (iii). Let $\alpha$ be an element of $\mathrm{Br}\, \kappa(X)$ and let $U$ be the complement in $X$ of the union of all components of the ramification locus that are of codimension 1 in $X$. Now, if $Z$ is any prime divisor on $U$, then $\alpha$ is unramified at some point of $Z$ and hence lies in $\mathrm{Br}\, \mathscr{O}_{U,Z}$. Applying (iii) to $U$, we see that $\alpha$ lies in $\mathrm{Br}\, U$. Thus the ramification locus of $\alpha$ contained only components of codimension 1, proving (iii) $\Rightarrow$ (i).

For the proof of the Purity Theorem, we refer to the literature. When $X$ has dimension at most 2, Proposition 7.2 of Auslander and Goldman (1960) shows that (iii) holds, proving the theorem in that case. In the general case, we use the fact that $\mathrm{Br}\, X$ is isomorphic to the étale cohomology group $\mathrm{H}^2(X, \mathbb{G}_\mathrm{m})$; see Proposition 12.4.13 for a discussion of this. The statement corresponding to (ii) for $\mathrm{H}^2(X, \mathbb{G}_\mathrm{m})$ is a deep result in étale cohomology, proved by Grothendieck (1968, III, Section 6). ☐

*Remark* 12.2.9 In positive characteristic $p$, the Purity Theorem is not known to hold in general for varieties of dimension greater than 2. However, the Brauer group is torsion **??** and so splits as the direct sum of a $p$-primary part and a prime-to-$p$ part. The Purity Theorem, and all the results below that depend on it, remain true for the prime-to-$p$ part of the Brauer group.

*Remark* 12.2.10 (The unramified Brauer group) For any field $K$ containing a base field $k$, we can consider the set $S$ of all discrete valuation rings $A \subset K$

containing $k$ and having fraction field $K$. For each such $A$, we have $\operatorname{Br} A \subset \operatorname{Br} K$, and so we define the *unramified Brauer group* of $K/k$ to be

$$\operatorname{Br}_{\mathrm{nr}}(K/k) = \bigcap_{A \in S} \operatorname{Br} A$$

where the intersection takes place in $\operatorname{Br} K$. In particular, this may be applied to the function field of a smooth, geometrically irreducible variety $X$ over $k$, giving us the unramified Brauer group $\operatorname{Br}_{\mathrm{nr}}(\kappa(X)/k)$. Let us compare this group, defined without any geometric information, to the Brauer group of $X$. Every prime divisor $Z$ on $X$ gives rise to a discrete valuation ring $O_{X,Z}$ that contains $k$ and has fraction field $\kappa(X)$; so any element of $\operatorname{Br}_{\mathrm{nr}}(\kappa(X)/k)$ is unramified at every divisor of $X$, and the Purity Theorem shows that $\operatorname{Br}_{\mathrm{nr}}(\kappa(X)/k)$ is contained in $\operatorname{Br} X$. On the other hand, not all discrete valuations on $\kappa(X)$ arise from prime divisors on $X$. For example, a nonempty open subset of $X$ has the same function field as $X$, but may have fewer prime divisors; and blowing up $X$ in a point gives a new variety with the same function field, but with one extra prime divisor. However, if $X$ is projective then it can be shown (using the valuative criterion of properness) that, for every discrete valuation ring $A$ as above, there is a point $x$ of $X$ satisfying $\mathscr{O}_{X,x} \subset A \subset \kappa(X)$. Therefore any element of $\operatorname{Br} X$ is contained in $\operatorname{Br} A$. So, when $X$ is projective, we have an identity $\operatorname{Br} X = \operatorname{Br}_{\mathrm{nr}}(\kappa(X)/k)$.

An important consequence of the Purity Theorem is that the Brauer group is a birational invariant of smooth, projective varieties. To show this, we first prove a lemma.

**Lemma 12.2.11** *Let $f\colon X \to Y$ be a birational morphism between smooth, geometrically irreducible varieties over a field $k$. Suppose that $Z \subset Y$ is a closed subset of codimension at least two such that the restriction $X \setminus f^{-1}(Z) \to Y \setminus Z$ is an isomorphism. If $Y$ satisfies the hypotheses of the Purity Theorem, then the induced map $f^*\colon \operatorname{Br} Y \to \operatorname{Br} X$ is an isomorphism.*

*Proof* We have a commutative diagram of Brauer groups as follows.

$$\begin{array}{ccc} \operatorname{Br} Y & \longrightarrow & \operatorname{Br}(Y \setminus Z) \\ {\scriptstyle f^*}\downarrow & & \downarrow \\ \operatorname{Br} X & \longrightarrow & \operatorname{Br}(X \setminus f^{-1}(Z)) \end{array}$$

The two horizontal arrows are inclusions; the top one is an equality, by the Purity Theorem. The right-hand vertical arrow is an isomorphism by assumption. Therefore the bottom arrow is an equality, and so $f^*$ is an isomorphism. □

*Remark* 12.2.12 Even in the absence of the Purity Theorem, the same argument shows that the map $f^*\colon \operatorname{Br} Y \to \operatorname{Br} X$ is injective.

**Corollary 12.2.13** *Let $\phi\colon X \dashrightarrow Y$ be a birational map of smooth, projective, geometrically irreducible surfaces over any field. Then the resulting isomorphism $\operatorname{Br} \kappa(Y) \cong \operatorname{Br} \kappa(X)$ induces an isomorphism $\operatorname{Br} Y \cong \operatorname{Br} X$.*

*Proof* By Theorem 7.2.2, we can factorise $\phi$ as $g \circ f^{-1}$, where $f\colon \tilde{X} \to X$ and $g\colon \tilde{X} \to Y$ are composites of blow-ups in points and isomorphisms. Because all the varieties concerned are surfaces, they satisfy the Purity Theorem. By Lemma 12.2.11, all the induced maps of Brauer groups are isomorphisms. □

*Remark* 12.2.14 One can prove in a similar way that the Brauer group is a birational invariant of smooth projective varieties of any dimension over a field of characteristic zero: see Grothendieck (1968, III, Section 7). Alternatively, this follows immediately from the identification of the Brauer group with the unramified Brauer group of the function field (see Remark 12.2.10), since the unramified Brauer group is clearly a birational invariant.

## 12.3 Examples

We are now in a position to compute the Brauer groups of some simple varieties.

**Theorem 12.3.1** *Let $k$ be an algebraically closed field, and let $C$ be a smooth, geometrically irreducible curve over $k$. Then $\operatorname{Br} C$ is trivial.*

*Proof* Corollary 10.4.10(ii) shows that $\operatorname{Br} \kappa(C)$ is trivial, and so the subgroup $\operatorname{Br} C$ is also trivial. □

In positive characteristic, the question of how the Brauer group changes when passing from a separably closed field to its algebraic closure was studied by Grothendieck using flat cohomology. In particular, he proved the following:

**Theorem 12.3.2** (Grothendieck) *Let $X$ be a projective variety over a separably closed field $k$, and denote by $\bar{X}$ the base change of $X$ to an algebraic closure of $k$. Suppose that the Picard variety of $X$ is smooth (for example, $\mathrm{H}^2(X, \mathscr{O}_X) = 0$ or $\mathrm{H}^1(X, \mathscr{O}_X) = 0$). Then the natural homomorphism $\operatorname{Br} X \to \operatorname{Br} \bar{X}$ is injective.*

*Proof* See Grothendieck (1968, Corollaire III.5.7). □

The hypothesis applies in particular to smooth, projective curves. Combining with Theorem 12.3.1, we obtain:

**Corollary 12.3.3** *Let $C$ be a smooth, geometrically irreducible, projective curve over a separably closed field. Then* $\operatorname{Br} C$ *is trivial.*

We next turn to computing the Brauer groups of affine and projective spaces.

**Lemma 12.3.4** *Let $k$ be any field. Then the only algebraic classes in* $\operatorname{Br}\mathbf{A}_k^1$ *are the constant classes, that is, the natural homomorphism* $\operatorname{Br} k \to \operatorname{Br}_1 \mathbf{A}_k^1$ *is an isomorphism.*

*Proof* This is the first part of Theorem 7.5 of Auslander and Goldman (1960). □

**Lemma 12.3.5** *Let $k$ be a perfect field. Then the natural homomorphism* $\operatorname{Br} k \to \operatorname{Br}\mathbf{A}_k^1$ *is an isomorphism.*

*Proof* Because $k$ is perfect, its separable closure is the same as its algebraic closure. Theorem 12.3.1 then shows that $\operatorname{Br}_1 \mathbf{A}_k^1$ is equal to $\operatorname{Br}\mathbf{A}_k^1$, and so Lemma 12.3.4 gives the desired result. □

As we noted in Example 12.1.11, if $k$ is not perfect then $\operatorname{Br}\mathbf{A}_k^1$ does contain non-constant elements.

**Theorem 12.3.6** *Let $k$ be a field of characteristic zero, and let $n$ be a positive integer. Then the natural homomorphism* $\operatorname{Br} k \to \operatorname{Br}\mathbf{A}_k^n$ *is an isomorphism.*

*Proof* We proceed by induction, following Colliot-Thélène (1980, Proposition 1.3). The case $n = 1$ follows from Lemma 12.3.5. Assume $n > 1$ and regard $\mathbf{A}_k^n$ as $\mathbf{A}_k^{n-1} \times \mathbf{A}_k^1$ with coordinates $x_1, \dots, x_{n-1}, t$. Let $\pi\colon \mathbf{A}_k^n \to \mathbf{A}_k^{n-1}$ be the projection onto the first $n-1$ coordinates, and let $\sigma\colon \mathbf{A}_k^{n-1} \to \mathbf{A}_k^n$ be the inclusion of the hyperplane $t = 0$. Thus $\pi \circ \sigma$ is the identity map on $\mathbf{A}_k^{n-1}$.

Let $K$ be the function field of $\mathbf{A}_k^{n-1}$, and $L$ the function field of $\mathbf{A}_k^n$; the map $\pi^*$ identifies $K$ with the subfield $k(x_1, \dots, x_{n-1})$ of $L = k(x_1, \dots, x_{n-1}, t)$. But $L$ is also the function field of the affine line over $K$, and so we have the Brauer group $\operatorname{Br}\mathbf{A}_K^1$ contained in $\operatorname{Br} L$. We claim that $\operatorname{Br}\mathbf{A}_k^n$ is contained in $\operatorname{Br}\mathbf{A}_K^1$. To see this, note that to every point $P$ of $\mathbf{A}_K^1$ is associated a subvariety of $\mathbf{A}_k^n$ (the scheme-theoretic closure of $P$), as follows: the point $P$ is defined by a prime ideal in $K[t]$, and the intersection of that prime ideal with the subring $k[x_1, \dots, x_{n-1}, t]$ is another prime ideal, which defines a subvariety $Z$ of $\mathbf{A}_k^n$. It is straightforward to check that the local rings $\mathscr{O}_{\mathbf{A}_K^1, P}$ and $\mathscr{O}_{\mathbf{A}_k^n, Z}$, both subrings of $L$, coincide. Thus any element of $\operatorname{Br} L$ which is unramified at $Z$ is also unramified at $P$, and so $\operatorname{Br}\mathbf{A}_k^n$ is contained in $\operatorname{Br}\mathbf{A}_K^1$. By Lemma 12.3.5, the homomorphism $\pi^*\colon \operatorname{Br} K \to \operatorname{Br} L$ maps $\operatorname{Br} K$ isomorphically to $\operatorname{Br}\mathbf{A}_K^1$; so $\operatorname{Br}\mathbf{A}_k^n$ is contained in the image of $\pi^*$.

Now let $\alpha$ be any class in $\operatorname{Br}\mathbf{A}_k^n$. As we have just seen, there is a class

$\beta \in \mathrm{Br}\, K$ satisfying $\alpha = \pi^*(\beta)$. But then we have $\sigma^*\alpha = \sigma^*\pi^*\beta = \beta$. By Proposition 12.2.3, $\beta$ lies in $\mathrm{Br}\,\mathbf{A}_k^{n-1}$. Thus the map $\pi^*\colon \mathrm{Br}\,\mathbf{A}_k^{n-1} \to \mathrm{Br}\,\mathbf{A}_k^n$ is surjective.

The natural map $\mathrm{Br}\, k \to \mathrm{Br}\,\mathbf{A}_k^n$ factors as $\mathrm{Br}\, k \to \mathrm{Br}\,\mathbf{A}_k^{n-1} \xrightarrow{\pi^*} \mathrm{Br}\,\mathbf{A}_k^n$. By the inductive hypothesis, the left-hand component is an isomorphism. We have shown that the right-hand component is surjective. The whole composite map is injective, since $\mathbf{A}_k^n$ has a $k$-point; so it is an isomorphism. □

**Corollary 12.3.7** *Let k be a field of characteristic zero, and let n be a positive integer. Then the natural homomorphism* $\mathrm{Br}\, k \to \mathrm{Br}\,\mathbf{P}_k^n$ *is an isomorphism.*

*Proof* Affine $n$-space $\mathbf{A}_k^n$ is an open subvariety of $\mathbf{P}_k^n$, so we have inclusions $\mathrm{Br}\, k \subseteq \mathrm{Br}\,\mathbf{P}_k^n \subseteq \mathrm{Br}\,\mathbf{A}_k^n$ inside $\mathrm{Br}\,\kappa(\mathbf{P}_k^n)$. By Theorem 12.3.6, all three groups coincide. □

The statement that $\mathrm{Br}\, k \to \mathrm{Br}\,\mathbf{P}_k^n$ is an isomorphism is true over any field, but in positive characteristic we cannot prove it in the same way as Corollary 12.3.7, for the simple reason that $\mathrm{Br}\,\mathbf{A}_k^n$ does contain non-constant elements. We describe an alternative approach by induction on $n$, again following Colliot-Thélène (1980, Proposition 1.3). Firstly, we prove the result for the projective line.

**Lemma 12.3.8** *Let k be any field. Then the natural map* $\mathrm{Br}\, k \to \mathrm{Br}\,\mathbf{P}_k^1$ *is an isomorphism.*

*Proof* By Corollary 12.3.3, $\mathrm{Br}_1\,\mathbf{P}_k^1$ is equal to $\mathrm{Br}\,\mathbf{P}_k^1$. But $\mathrm{Br}_1\,\mathbf{P}_k^1$ is contained in $\mathrm{Br}_1\,\mathbf{A}_k^1$, which is the image of $\mathrm{Br}\, k$ by Lemma 12.3.4. □

Now we can apply Lemma 12.3.8 inductively in the same way as in the proof of Theorem 12.3.6 to prove the result for $\mathbf{P}^n$.

**Theorem 12.3.9** *Let k be any field, and let n be a positive integer. Then the natural homomorphism* $\mathrm{Br}\, k \to \mathrm{Br}\,\mathbf{P}_k^n$ *is an isomorphism.*

*Proof* Suppose that $n$ is greater than 1. Pick a point $P$ of $\mathbf{P}_k^n$, and let $p\colon \mathbf{P}_k^n \dashrightarrow \mathbf{P}_k^{n-1}$ be the projection away from the point $P$. This rational map is not defined at $P$. However, $p$ factors as $\mathbf{P}_k^n \xleftarrow{f} X \xrightarrow{\pi} \mathbf{P}_k^{n-1}$ where $f$ is the blow-up of $\mathbf{P}^n$ at the point $P$ and $\pi$ is a morphism. By Remark 12.2.12, the induced map $f^*\colon \mathrm{Br}\,\mathbf{P}_k^n \to \mathrm{Br}\, X$ is injective, so it suffices to show that $\mathrm{Br}\, k \to \mathrm{Br}\, X$ is an isomorphism. This follows from Colliot-Thélène (1980, Proposition 1.3), as before; we sketch the details.

Notice that $\pi$ is a morphism having fibres isomorphic to $\mathbf{P}_k^1$; and the inclusion $\sigma$ of the exception divisor of the blow-up in $X$ is a section of $\pi$. So we are in a situation very much like that of Theorem 12.3.6. Letting $K$ denote the function field of $\mathbf{P}_k^{n-1}$, we again have an identification of the function field $L$ of $X$ with the function field of $\mathbf{P}_K^1$. (Indeed, the function field of $X$ is the same as that of $\mathbf{A}^n$, so the identification is the same as in the proof of Theorem 12.3.6.) Under this identification, $\operatorname{Br} X$ is contained in $\operatorname{Br} \mathbf{P}_K^1$; as before, this is true because every divisor on $\mathbf{P}_K^1$ can be identified with a divisor on $X$. By Lemma 12.3.8, the homomorphism $\pi^* \colon \operatorname{Br} K \to \operatorname{Br} L$ maps $\operatorname{Br} K$ isomorphically to $\operatorname{Br} \mathbf{P}_K^1$, and therefore $\operatorname{Br} X$ is contained in the image of $\pi^*$.

Finally, we use the section $\sigma$ in the same way as in the proof of Theorem 12.3.6 to deduce that $\pi^*$ maps $\operatorname{Br} \mathbf{P}_k^{n-1}$ isomorphically to $\operatorname{Br} X$. By induction, the natural homomorphism $\operatorname{Br} k \to \operatorname{Br} X$ is an isomorphism, and the result follows. □

**Corollary 12.3.10** *Let $k$ be any field, and let $X$ be a smooth, projective, geometrically rational surface over $k$. Then there is an equality* $\operatorname{Br}_1 X = \operatorname{Br} X$.

*Proof* Let $\bar{k}$ be an algebraic closure of $k$, and $k^{\mathrm{sep}}$ the separable closure of $k$ in $\bar{k}$. By assumption, the base change $\bar{X}$ of $X$ to $\bar{k}$ is birational to $\mathbf{P}_{\bar{k}}^2$. By Corollary 12.2.13, there is an isomorphism $\operatorname{Br} \bar{X} \cong \operatorname{Br} \mathbf{P}_{\bar{k}}^2$. By Corollary 12.3.7, we have $\operatorname{Br} \mathbf{P}_{\bar{k}}^2 \cong \operatorname{Br} \bar{k} = 0$. If $k$ has characteristic zero, then this completes the proof.

If $k$ has positive characteristic, we must work a little harder. We have $\mathrm{H}^1(X, \mathscr{O}_X) = 0$, since coherent sheaf cohomology groups respect base change and $\mathrm{H}^1(\bar{X}, \mathscr{O}_{\bar{X}})$ is trivial because $\bar{X}$ is rational. Therefore Theorem 12.3.2 applies, and we deduce that $\operatorname{Br} X^{\mathrm{sep}}$ is trivial. It follows that $\operatorname{Br}_1 X = \ker(\operatorname{Br} X \to \operatorname{Br} X^{\mathrm{sep}})$ is the whole of $\operatorname{Br} X$. □

We will see how to compute $\operatorname{Br}_1 X$ using Galois cohomology, in Chapter 15.

## 12.4 Other definitions of the Brauer group

The definition of the Brauer group in Section 12.1 is convenient: by defining the Brauer group of a smooth, geometrically irreducible variety to be a subgroup of the Brauer group of its function field, we can apply all our knowledge of Brauer groups of fields to understand the Brauer group of a variety. In particular, our definition is useful for computations, as we will see in Chapter 15. However, the definition is also lacking in some respects: most notably, it does not generalise well to singular varieties, nor to those which are reducible. For deeper study of Brauer groups, it is important to be able to talk about the Brauer group of an arbitrary variety, or indeed an arbitrary scheme. In this section we describe two more general notions of the Brauer group: one defined in terms of sheaves of Azumaya algebras, and the other using étale cohomology. For a smooth, geometrically irreducible, quasi-projective variety over a field, both these notions coincide with the definition of Section 12.1. The material of this section is not necessary for understanding any subsequent parts of this book.

In this section, we will freely use schemes, sheaves and étale cohomology. Most of the material in this section is covered in detail by Milne (1980) and by Grothendieck (1968). We reserve the notation $\mathrm{Br}\, X$ for the Brauer group as defined in Definition 12.1.1.

It often happens in algebraic geometry that classes of objects defined over a base ring may be extended without much work to classes of objects over an arbitrary scheme. This can be done to the definition of an Azumaya algebra, giving the notion of an Azumaya algebra over a scheme (or, indeed, over a ringed space). This definition was first made by Auslander (1966).

**Proposition 12.4.1** *Let $X$ be a scheme, and let $\mathscr{A}$ be a locally free sheaf of $\mathscr{O}_X$-algebras of finite presentation. The following are equivalent:*

(i) **A** *has non-zero rank everywhere, and the natural map*

$$\phi\colon \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{A}^{\mathrm{opp}} \to \mathscr{E}nd_{\mathscr{O}_X} \mathscr{A},$$

*defined on an open set $U$ by $(a \otimes a') \mapsto (x \mapsto axa')$, is an isomorphism;*

(ii) *for every point $x \in X$, the stalk $\mathscr{A}_x$ is an Azumaya algebra over the local ring $\mathscr{O}_{X,x}$;*

(iii) *for every point $x \in X$, the fibre $\mathscr{A}_x \otimes_{\mathscr{O}_{X,x}} k_x$ is a central simple algebra over the residue field $k_x$;*

(iv) *there is an étale morphism $f\colon X' \to X$ such that $f^*\mathscr{A}$ is isomorphic to the matrix algebra $\mathrm{M}_n(\mathscr{O}_X)$, for some positive integer $n$.*

*Proof* See Milne (1980, Chapter IV, Proposition 2.1). □

**Definition 12.4.2** Let $X$ be a scheme. An *Azumaya algebra* on $X$ is a locally free, finitely presented sheaf of $\mathscr{O}_X$-algebras satisfying the equivalent properties of Proposition 12.4.1.

In the case that $X = \mathrm{Spec}\, R$ is an affine scheme, there is a standard equivalence between $R$-modules and sheaves of $\mathscr{O}_X$-modules. Proposition 11.2.3 shows that an $R$-algebra $A$ is an Azumaya algebra (in the sense of Chapter 11) if and only if the corresponding sheaf $\tilde{A}$ is an Azumaya algebra on $X$.

We can also sheafify the definition of equivalence of Azumaya algebras.

**Definition 12.4.3** Let $X$ be a scheme. Two Azumaya algebras $\mathscr{A}, \mathscr{B}$ on $X$ are *equivalent* if there are locally free sheaves of $\mathscr{O}_X$-modules $\mathscr{E}, \mathscr{E}'$, everywhere of finite non-zero rank, such that $\mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{E}nd\mathscr{E}$ is isomorphic to $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{E}nd\mathscr{E}'$.

Again, this definition generalises that of Chapter 11 in the case that $X = \operatorname{Spec} R$ is affine.

**Definition 12.4.4** Let $X$ be a scheme. The *Azumaya Brauer group* of $X$, denoted $\mathrm{Br}_{\mathrm{Az}} X$, is the group of equivalence classes of Azumaya algebras over $X$, with the operation induced by tensor product.

In order to study the Azumaya Brauer group of a scheme, and in particular to show that it agrees with our Brauer group for smooth, geometrically irreducible varieties, we would like to relate it to some cohomology groups of $X$. From a formal point of view, this proceeds in exactly the same way as the identification of the Brauer group of a field with the second Galois cohomology group. Proposition 12.4.1(iv) shows that an Azumaya algebra of constant rank on $X$ is a twisted form of the matrix algebra $\mathrm{M}_n(\mathscr{O}_X)$ for some $n$.

**Proposition 12.4.5** *For any scheme $X$, there is a natural injection* $\mathrm{Br}_{\mathrm{Az}} X \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$.

*Sketch of proof* See Milne (1980, Theorem 2.5) for the details. We sketch the argument in the case when Čech cohomology on $X_{\text{ét}}$ agrees with derived-functor cohomology, to point out the analogy with the proof of Theorem **??**.

Firstly, a sheafified version of the Skolem–Noether theorem (Theorem 10.3.12) shows that the automorphism sheaf of $\mathrm{M}_n(\mathscr{O}_X)$ is the sheaf associated to the group scheme $\mathrm{PGL}_n$. Then standard Čech cohomology theory gives an injection from the set of isomorphism classes of Azumaya algebras of rank $n^2$ on $X$ to the set $\mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n)$, which one can show is actually a bijection.

The exact sequence of sheaves on $X_{\text{ét}}$

$$0 \to \mathbb{G}_{\mathrm{m}} \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 0$$

gives rise to a long exact sequence in (non-abelian) cohomology, part of which is

$$\mathrm{H}^1_{\text{ét}}(X, \mathrm{GL}_n) \to \mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n) \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}}).$$

The first group $\mathrm{H}^1_{\text{ét}}(X, \mathrm{GL}_n)$ parametrises isomorphism classes of locally free $O_X$-modules of rank $n$ on $X$. A calculation shows that the homomorphism $\mathrm{H}^1_{\text{ét}}(X, \mathrm{GL}_n) \to \mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n)$ takes the class of a vector bundle $\mathscr{E}$ to the class of the endomorphism algebra $\mathscr{E}nd\mathscr{E}$. Thus the kernel of the map $\mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n) \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$ consists of isomorphism classes of trivial Azumaya algebras, and we obtain an embedding

$$\{\text{equivalence classes of Azumaya algebras of rank } n^2\} \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_\mathrm{m}).$$

It can then be shown that these maps for individual $n$ together induce an injection $\mathrm{Br}_{\mathrm{Az}} X \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_\mathrm{m})$. □

Motivated by Proposition 12.4.5, we define a new variant of the Brauer group of a scheme.

**Definition 12.4.6** Let $X$ be a scheme. The *cohomological Brauer group* of $X$, denoted $\mathrm{Br}' X$, is the étale cohomology group $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_\mathrm{m})$.

Beware that some authors use $\mathrm{Br} X$ to denote the Azumaya Brauer group, and some use $\mathrm{Br} X$ to denote the cohomological Brauer group. Also, some authors use the name "cohomological Brauer group" and the notation $\mathrm{Br}' X$ or $\mathrm{Br} X$ to refer to the torsion subgroup of what we have called $\mathrm{Br}' X$.

In the case of a field $K$, the group $\mathrm{Br}'(K)$ is equal to the Galois cohomology group $\mathrm{H}^2(K, K^\times)$ and, as we saw in **??**, the injection $\mathrm{Br} K \to \mathrm{Br}' K$ is an isomorphism. It is a natural question to ask whether the equality $\mathrm{Br}_{\mathrm{Az}} = \mathrm{Br}'$ holds in more generality. Before turning to this question, we look at the relationship between $\mathrm{Br}' X$ and $\mathrm{Br}\, \kappa(X)$.

**Proposition 12.4.7** *Let $X$ be a regular integral Noetherian scheme. Then the natural map* $\mathrm{Br}' X \to \mathrm{Br}' \kappa(X) = \mathrm{Br}\, \kappa(X)$ *is injective.*

*Proof* We sketch the proof of Grothendieck (1968, Section II.1); see also Milne (1980, Chapter III, Example 2.22). Let $j\colon \mathrm{Spec}\, \kappa(X) \to X$ denote the inclusion of the generic point. The short exact sequence of sheaves on $X_{\text{ét}}$

$$0 \to \mathbb{G}_\mathrm{m} \to j_* \mathbb{G}_\mathrm{m} \to \mathscr{D}iv_X \to 0$$

defines the sheaf $\mathscr{D}iv_X$ of Cartier divisors on $X$. From it we deduce an exact sequence

$$0 \to \mathrm{H}^1_{\text{ét}}(X, \mathscr{D}iv_X) \to \mathrm{Br}' X \to \mathrm{H}^2_{\text{ét}}(X, j_* \mathbb{G}_\mathrm{m})$$

of cohomology groups. A calculation using Hilbert's Theorem 90 shows that $\mathrm{R}^1 j_* \mathbb{G}_\mathrm{m}$ vanishes. The Leray spectral sequence for $j$ then shows that $\mathrm{H}^2_{\text{ét}}(X, j_* \mathbb{G}_\mathrm{m})$ injects into $\mathrm{Br}\, \kappa(X)$, and so we deduce that the kernel of the map $\mathrm{Br}' X \to \mathrm{Br}\, \kappa(X)$ is given by $\mathrm{H}^1_{\text{ét}}(X, \mathscr{D}iv_X)$. Because $X$ is regular, the Cartier divisors on $X$ are the same as the Weil divisors; thus there is an isomorphism of sheaves $\mathscr{D}iv_X \cong \bigoplus_D (i_D)_* \mathbf{Z}$, where $\imath_D\colon D \to X$ runs over the inclusions of all prime divisors on $X$. We deduce an isomorphism $\mathrm{H}^1_{\text{ét}}(X, \mathscr{D}iv_X) \cong \bigoplus_D \mathrm{H}^1_{\text{ét}}(D, \mathbf{Z})$, and an easy calculation show that the groups $\mathrm{H}^1_{\text{ét}}(D, \mathbf{Z})$ are trivial. Hence $\mathrm{Br}' X$ injects into $\mathrm{Br}\, \kappa(X)$. □

The proof of Proposition 12.4.7 also shows how to construct varieties $X$ such that $\mathrm{Br}' X \to \mathrm{Br}\,\kappa(X)$ is *not* injective. For example, let $X$ be an integral normal surface over an algebraically closed field. Developing the arguments of Grothendieck (1968, Section II.1), DeMeyer and Ford (1992) show that the kernel of the map $\mathrm{Br}' X \to \mathrm{Br}\,\kappa(X)$ is isomorphic to the cokernel of the natural map

$$\mathrm{Cl}(X) \to \bigoplus_P \mathrm{Cl}(\mathscr{O}^h_{X,P})$$

where Cl denotes the Weil divisor class group, $P$ runs over the singular points of $X$, and $\mathscr{O}^h_{X,P}$ denotes the Henselisation of the local ring at $P$. Grothendieck refers to a construction of Mumford giving a variety for which this quotient is non-zero, and even non-torsion. Childs (1976) gives several explicit examples of this kind.

**Example 12.4.8** (Childs, 1976, Theorem 5.1) Let $X$ be the affine variety over the complex numbers $\mathbf{C}$ defined by $z^2 = ux^2 + vy^2$, where $u, v$ are polynomials in $x, y$ not vanishing at $(0,0)$, and let $P$ be the singular point $(0,0,0)$. Then $\mathrm{Cl}(\mathscr{O}^h_{X,P})$ has order 2; and $\mathrm{Cl}(X)$ has order 2 if and only if the equation $1 = u\alpha^2 + v\beta^2$ has a solution in the local ring $\mathscr{O}_{X,P}$, and has order 1 otherwise. Thus the kernel of $\mathrm{Br}' X \to \mathrm{Br}\,\kappa(X)$ has order 1 or 2, accordingly.

**Example 12.4.9** (Childs, 1976, Theorem 6.1) Let $C$ be a smooth plane curve over the complex numbers $\mathbf{C}$ of genus greater than three. Let $X$ be the affine cone over $C$. Then $\ker(\mathrm{Br}' X \to \mathrm{Br}\,\kappa(X))$ is a complex vector space of positive dimension.

For examples of singular del Pezzo surfaces $X$ where $\ker(\mathrm{Br}' X \to \mathrm{Br}\,\kappa(X))$ is non-trivial, see Bright (2013).

Let us return to the question of when the Azumaya Brauer group $\mathrm{Br}_{\mathrm{Az}}$ coincides with the cohomological Brauer group $\mathrm{Br}'$. One possible obstruction is given by the following lemma.

**Lemma 12.4.10** *Let $X$ be a scheme with finitely many connected components. The Azumaya Brauer group $\mathrm{Br}_{\mathrm{Az}}(X)$ is a torsion group.*

*Proof* The image of $\mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n)$ in $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$ is killed by $n$ (see Milne, 1980, Chapter IV, Proposition 2.7), and so any Azumaya algebra of rank $n^2$ corresponds to class of order dividing $n$ in $\mathrm{Br}_{\mathrm{Az}} X$. The rank of a locally free $\mathscr{O}_X$-module is locally constant on $X$, so the lemma is true if $X$ is connected. In the general case, the Azumaya Brauer group of $X$ is the product of those of its connected components, and a product of finitely many torsion groups is again torsion. □

In particular, this shows that any scheme $X$ such that $\mathrm{Br}' X$ is non-torsion (for example, the surfaces of Example 12.4.9) cannot possibly satisfy $\mathrm{Br}_{\mathrm{Az}} X = \mathrm{Br}' X$. However, the following theorem shows that, for quasi-projective varieties, this is the only obstruction.

**Theorem 12.4.11** *Let $X$ be a scheme admitting an ample invertible sheaf. Then the inclusion $\mathrm{Br}_{\mathrm{Az}} X \to \mathrm{Br}' X$ identifies $\mathrm{Br}_{\mathrm{Az}} X$ with the torsion subgroup of $\mathrm{Br}' X$.*

*Proof* This theorem is originally due to Gabber (unpublished). An alternative proof has been given by de Jong (n.d.). □

For a discussion of the proof of this theorem in the case that $X$ is affine, or the union of two affine subschemes, see Hoobler (1982).

Finally, let us show that the definition of the Brauer group of a smooth, geometrically irreducible variety $X$ over a field given in Definition 12.1.1 does indeed coincide with both $\mathrm{Br}_{\mathrm{Az}} X$ and $\mathrm{Br}' X$. As noted above, if $X$ is affine then $\mathrm{Br}_{\mathrm{Az}} X$ coincides with our $\mathrm{Br} X$ almost by definition.

**Lemma 12.4.12** *Let $X$ be a regular integral scheme. Then $\mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{O}_X^\times)$ is trivial for all $p > 1$.*

*Proof* Let $R_X$ be the sheaf of rational functions on $X$. We have the Weil-divisor exact sequence

$$0 \to \mathscr{O}_X^\times \to R_X^\times \to \bigoplus_{i_D \colon D \to X} (i_D)_* \mathbf{Z} \to 0$$

where the direct product is over all inclusions $i_D \colon D \to X$ of prime divisors on $X$. The sheaves $R_X^\times$ and $\bigoplus_D (i_D)_* \mathbf{Z}$ are both flasque, so this sequence is a flasque resolution of $\mathscr{O}_X^\times$ and can be used to compute its cohomology. Therefore $\mathrm{H}^p_{\mathrm{Zar}}(X, \mathscr{O}_X^\times)$ vanishes for $p > 1$. □

**Proposition 12.4.13** *Let $X$ be a smooth, geometrically irreducible variety over a field. If we identify $\mathrm{Br}' X = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{\mathrm{m}})$ with its image in $\mathrm{Br}\, \kappa(X)$, then the Brauer group $\mathrm{Br} X$ of Definition 12.1.1 coincides with $\mathrm{Br}' X$.*

*Proof* Let $\mathscr{H}^i(\mathbb{G}_{\mathrm{m}})$ denote the Zariski sheaf on $X$ associated to the presheaf defined by $U \mapsto \mathrm{H}^i_{\text{ét}}(U, \mathbb{G}_{\mathrm{m}})$. We claim that our $\mathrm{Br} X$ is naturally identified with $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathscr{H}^2(\mathbb{G}_{\mathrm{m}}))$, as follows. The description of sheafification given by Hartshorne (1977, p. 64) shows that a section of $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathscr{H}^2(\mathbb{G}_{\mathrm{m}}))$ is given by a locally compatible family of elements of the stalks $\mathrm{Br}'(\mathrm{Spec}\, \mathscr{O}_{X,x})$; given that these stalks all inject into $\mathrm{Br}\, \kappa(X)$, it follows that $\mathrm{H}^0_{\mathrm{Zar}}(X, \mathscr{H}^2(\mathbb{G}_{\mathrm{m}}))$ is identified with the intersection in $\mathrm{Br}\, \kappa(X)$ of the subgroups $\mathrm{Br}'(\mathrm{Spec}\, \mathscr{O}_{X,x})$. By Theorem 12.4.11 we have $\mathrm{Br}'(\mathrm{Spec}\, \mathscr{O}_{X,x}) = \mathrm{Br}_{\mathrm{Az}}(\mathrm{Spec}\, \mathscr{O}_{X,x}) = \mathrm{Br}\, \mathscr{O}_{X,x}$, compatibly with the inclusions in $\operatorname{Br}\kappa(X)$, and so indeed the natural injection of $\mathrm{H}^0_{\mathrm{Zar}}(X,\mathscr{H}^2(\mathbb{G}_\mathrm{m}))$ into $\operatorname{Br}\kappa(X)$ identifies it with $\operatorname{Br}X$.

The sheaf $\mathscr{H}^i(\mathbb{G}_\mathrm{m})$ coincides with the higher direct image sheaf $\mathrm{R}^i f_*\mathbb{G}_\mathrm{m}$, where $f\colon X_{\text{ét}} \to X_{\mathrm{Zar}}$ is the natural map of sites. The Leray spectral sequence for $f$ is

$$E_2^{pq} = \mathrm{H}^p_{\mathrm{Zar}}(X,\mathscr{H}^q(\mathbb{G}_\mathrm{m})) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X,\mathbb{G}_\mathrm{m}).$$

The sheaf $\mathscr{H}^1(\mathbb{G}_\mathrm{m})$ is the zero sheaf: if $x$ is a closed point of $X$, then the stalk of $\mathscr{H}^1(\mathbb{G}_\mathrm{m})$ at $x$ is the group $\operatorname{Pic}(\operatorname{Spec}\mathscr{O}_{X,x})$, which vanishes. Directly from the definition, $\mathscr{H}^0(\mathbb{G}_\mathrm{m})$ is the Zariski sheaf $\mathscr{O}_X^\times$, and so $\mathrm{H}^p_{\mathrm{Zar}}(X,\mathscr{H}^0(\mathbb{G}_\mathrm{m}))$ vanishes for $p > 1$ by Lemma 12.4.12. The spectral sequence therefore shows that the natural map $\operatorname{Br}' X \to \mathrm{H}^0_{\mathrm{Zar}}(X,\mathscr{H}^2(\mathbb{G}_\mathrm{m}))$ is an isomorphism, which by functoriality is compatible with the natural embeddings of both groups in $\operatorname{Br}\kappa(X)$. □

- Say something about Brauer groups over the complex numbers (using exponential sequence from Chapter 2?).
- Brauer groups of some varieties (e.g. finiteness of Brauer groups of K3 surfaces)

# 13

# The Brauer–Manin obstruction

Manin (1971) realised that we can sometimes use an element of the Brauer group of a variety $X$ to show that some adelic points on $X$ cannot possibly be approximations to rational points; in extreme cases, this construction shows that $X$ has no rational points. This obstruction to the existence of rational points is now known as the Brauer–Manin obstruction. It puts many counter-examples to the Hasse principle, such as those we saw in Chapter 2, into a general framework.

## 13.1 The obstruction

Throughout this section, let $X$ be a smooth, projective, geometrically irreducible variety over a number field $k$.

Let $\ell$ be a field containing $k$. Recall that, given an element $\alpha$ of $\operatorname{Br} X$ and a point $P \in X(\ell)$, we can evaluate $\alpha$ at $P$ to get a class $\alpha(P) \in \operatorname{Br} \ell$. In particular, for each place $v$ of $k$, we can evaluate $\alpha \in \operatorname{Br} X$ at points of $X(k_v)$ to get classes in $\operatorname{Br} k_v$. We denote by $\alpha$ also the evaluation map $X(k_v) \to \operatorname{Br} k_v$ thus obtained. Composing with the local invariant map gives a *local evaluation map*

$$X(k_v) \xrightarrow{\alpha} \operatorname{Br} k_v \xrightarrow{\operatorname{inv}_v} \mathbf{Q}/\mathbf{Z} \tag{13.1}$$

for each place $v$. Before we can describe the Brauer–Manin obstruction, we need a finiteness result about these maps.

**Proposition 13.1.1** *Let $\alpha$ be a class in* $\operatorname{Br} X$*. Then, for all but finitely many places $v \in \Omega_k$, we have $\alpha(P) = 0$ for all $P \in X(k_v)$.*

The proof of Proposition 13.1.1 is a classic "spreading out" type proof, similar to that of Lemma 2.2.3. In the language of schemes it can be made very brief (see Skorobogatov, 2001, p. 101) but we can also present the proof in a more elementary way. We first prove it for integral points on an affine variety. Recall that, for an affine variety $Y \subset \mathbf{A}^n$ over a number field $k$, and a place $v$, the notation $Y(\mathfrak{o}_v)$ simply means the set of points of $Y(k_v)$ having coordinates that all lie in $\mathfrak{o}_v$.

**Lemma 13.1.2** *Let $k$ be a number field. Let $Y \subset \mathbf{A}^n_k$ be a smooth, geometrically irreducible variety over $k$, and let $\alpha$ be a class in $\operatorname{Br} Y$. Then, for all but finitely many places $v \in \Omega_k$, we have $\alpha(P) = 0$ for all $P \in Y(\mathfrak{o}_v)$.*

*Proof* Let $\mathfrak{o}_k$ be the ring of integers of $k$. Let $I$ be the intersection of the ideal $I(X,k) \subset k[x_1,\dots,x_n]$ with the subring $\mathfrak{o}_k[x_1,\dots,x_n]$, and let $R$ be the quotient ring $\mathfrak{o}_k[x_1,\dots,x_n]/I$. It is a subring of the coordinate ring $k[Y] = k[x_1,\dots,x_n]/I(X,k)$, and we have $k[Y] = k.R$. In particular, this shows that $R$ is an integral domain. (In scheme-theoretic terms, $\operatorname{Spec} R$ is the Zariski closure of $Y$ in $\mathbf{A}^n_{\mathfrak{o}_k}$.)

For every point $P \in Y(\bar{k})$, we have $\alpha \in \operatorname{Br} \mathscr{O}_{Y,P} \subset \operatorname{Br} \kappa(Y)$. If $\mathfrak{p} \subset R$ denotes the prime ideal that is the kernel of evaluation at $P$, then the local ring $\mathscr{O}_{Y,P}$ is also the localisation of $R$ at $\mathfrak{p}$. Applying Corollary 11.3.9 shows that there exists $f_P \in R$ satisfying $f_P(P) \neq 0$, and such that $\alpha$ lies in the image of $\operatorname{Br} R[f_P^{-1}] \to \operatorname{Br} \mathscr{O}_{Y,P}$. Let $J \subset k[Y]$ be the ideal generated by all the $f_P$ for $P \in Y(\bar{k})$. By the Nullstellensatz, $J$ is the unit ideal. Thus we can write

$$a_{P_1} f_{P_1} + \cdots + a_{P_r} f_{P_r} = 1$$

where $P_1,\dots,P_r \in Y(\bar{k})$ are finitely many points, each $a_{P_i}$ lies in $k[Y]$, and $\alpha$ lies in the image of $\operatorname{Br} R[f_{P_i}^{-1}]$ for all $i$. Clearing denominators gives

$$a'_{P_1} f_{P_1} + \cdots + a'_{P_r} f_{P_r} = N$$

with $a'_{P_i} \in R$ and $N \in \mathfrak{o}_k$. We claim that the conclusion follows for all finite places $v$ satisfying $v(N) = 0$.

If $v$ is a finite place with $v(N) = 0$, then let $P$ lie in $Y(\mathfrak{o}_v)$. Substituting $P$ into the above sum shows that some $f = f_{P_i}$ satisfies $v(f(P)) = 0$. Thus the image of $f$ under the evaluation map $\mathrm{ev}_P \colon R \to \mathfrak{o}_v$ is invertible, and so the evaluation map factors through $R[f^{-1}]$. We obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{Br} R[f^{-1}] & \longrightarrow & \operatorname{Br} \mathscr{O}_{Y,P} \\ \downarrow{\scriptstyle \mathrm{ev}_P} & & \downarrow{\scriptstyle \mathrm{ev}_P} \\ \operatorname{Br} \mathfrak{o}_v & \longrightarrow & \operatorname{Br} k_v. \end{array}$$

Because $\alpha$ lies in the image of $\operatorname{Br} R[f^{-1}]$ in $\operatorname{Br} \mathscr{O}_{Y,P}$, so $\alpha(P)$ lies in the image of $\operatorname{Br} \mathfrak{o}_v$ in $\operatorname{Br} k_v$. But $\operatorname{Br} \mathfrak{o}_v$ is trivial by Corollary 11.3.13. □

*Proof of Proposition 13.1.1* Applying Lemma 13.1.2 to each of the standard affine pieces of $X$ gives a finite set $S \subset \Omega_k$ such that, for $v \notin S$, the algebra $\alpha$ evaluates to zero on all $\mathfrak{o}_v$-points of all affine pieces. Since $X$ is projective, every $k_v$-point of $X$ can be scaled to give an $\mathfrak{o}_v$-point on one of the standard affine pieces of $X$. □

*Remark* 13.1.3 The condition that $X$ be projective cannot be removed from Proposition 13.1.1. Indeed, Harari (1994, Théorème 2.1.1) has shown the following: if $U \subset X$ is a non-empty open subset of a smooth, projective, geometrically irreducible variety over a number field $k$, and $\alpha \in \operatorname{Br} U$ is a class which does not lie in $\operatorname{Br} X$, then there are infinitely many places $v$ of $k$ where the local evaluation map $U(k_v) \to \operatorname{Br} k_v$ associated to $\alpha$ is not zero.

Because $X$ is projective, the set of adelic points $X(\mathbf{A}_k)$ of $X$ is simply the direct product $\prod_v X(k_v)$. Fix a class $\alpha \in \operatorname{Br} X$. Adding together all the local maps (13.1), we obtain an adelic evaluation map

$$X(\mathbf{A}_k) \to \bigoplus_v \operatorname{Br} k_v \xrightarrow{\sum_v \operatorname{inv}_v} \mathbf{Q}/\mathbf{Z}. \tag{13.2}$$

For each place $v$ of $k$, the set $X(k)$ is a subset of $X(k_v)$. Combining all the places, we obtain a diagonal embedding of $X(k)$ in the set $X(\mathbf{A}_k)$ of adelic points. The following observation is the key to the definition of the Brauer–Manin obstruction.

**Proposition 13.1.4** *Let $\alpha$ be a class in* $\operatorname{Br} X$. *Then $X(k)$ lies in the kernel of the map* (13.2).

*Proof* It is straightforward to check that the following diagram commutes:

$$\begin{array}{ccccc} X(k) & \longrightarrow & X(\mathbf{A}_k) & & \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} & & \\ \operatorname{Br} k & \longrightarrow & \bigoplus_v \operatorname{Br} k_v & \xrightarrow{\sum_v \operatorname{inv}_v} & \mathbf{Q}/\mathbf{Z} \end{array} \tag{13.3}$$

where the vertical arrows are evaluation of $\alpha$ at points, the top horizontal arrow is the inclusion of $X(k)$ in $X(\mathbf{A}_k)$, and the bottom line is the exact sequence (**??**). The composite map from $X(\mathbf{A}_k)$ to $\mathbf{Q}/\mathbf{Z}$ is the map of (13.2). Now suppose that $x \in X(k)$ is a point of $X$. Then we have $\sum_v \operatorname{inv}_v \alpha(x) = 0$ because $\alpha(x)$ lies in $\operatorname{Br} k$, and the bottom row of (13.3) is a complex. Hence the diagonal image of $x$ in $X(\mathbf{A}_k)$ lies in the kernel of the adelic evaluation map, as claimed. □

With Proposition 13.1.4 in mind, we make the following definition.

**Definition 13.1.5** Let $\alpha$ be a class in $\mathrm{Br}X$. Define

$$X(\mathbf{A}_k)^\alpha := \Big\{(P_v) \in X(\mathbf{A}_k) \;\Big|\; \sum_v \mathrm{inv}_v\, \alpha(P_v) = 0\Big\}.$$

If $B$ is a subset of $\mathrm{Br}X$, similarly define

$$X(\mathbf{A}_k)^B := \Big\{(P_v) \in X(\mathbf{A}_k) \;\Big|\; \sum_v \mathrm{inv}_v\, \alpha(P_v) = 0 \text{ for all } \alpha \in B\Big\};$$

we use the notation $X(\mathbf{A}_k)^{\mathrm{Br}}$ for $X(\mathbf{A}_k)^{\mathrm{Br}X}$.

One way to look at this is as follows: the map (13.2) defines a pairing $X(\mathbf{A}_k) \times \mathrm{Br}X \to \mathbf{Q}/\mathbf{Z}$, and we have defined $X(\mathbf{A}_k)^B$ to be the subset of $X(\mathbf{A}_k)$ orthogonal to the set $B$ under this pairing.

Proposition 13.1.4 states that $X(k)$ is contained in $X(\mathbf{A}_k)^B$ for any subset $B$ of $\mathrm{Br}X$. In particular, if $X(\mathbf{A}_k)^B$ is empty, then $X(k)$ is also empty.

**Definition 13.1.6** Let $X$ be a smooth, projective, geometrically irreducible variety over a number field $k$. Let $B$ be a subset of the Brauer group of $X$. If $X(\mathbf{A}_k)$ is not empty and $X(\mathbf{A}_k)^B$ is empty, then we say that there is a *Brauer–Manin obstruction to the Hasse principle* on $X$ coming from $B$. If $X(\mathbf{A}_k)^B$ is strictly contained in $X(\mathbf{A}_k)$, we say that there is a *Brauer–Manin obstruction to weak approximation* on $X$ coming from $B$. If $B$ equals $\mathrm{Br}X$, we simply say that there is a Brauer–Manin obstruction to the Hasse principle or to weak approximation on $X$.

Notice that constant classes in $\mathrm{Br}X$ give no contribution to the obstruction.

**Proposition 13.1.7** *If $\alpha \in \mathrm{Br}X$ lies in $\mathrm{Br}_0 X$, then the associated map* (13.2) *is zero.*

*Proof* This follows immediately from the exact sequence (**??**). □

It is a common abuse of notation to write $\mathrm{Br}X/\mathrm{Br}\,k$ for the quotient $\mathrm{Br}X/\mathrm{Br}_0 X$, even in the case in which the map $\mathrm{Br}\,k \to \mathrm{Br}X$ is not injective.

*Remark* 13.1.8 In view of Proposition 13.1.7, the pairing $X(\mathbf{A}_k) \times \mathrm{Br}X \to \mathbf{Q}/\mathbf{Z}$ is well defined when $\mathrm{Br}X$ is replaced by $\mathrm{Br}X/\mathrm{Br}\,k$. In many of the cases of interest, $\mathrm{Br}X/\mathrm{Br}\,k$ is a finite group, and it is possible to calculate $X(\mathbf{A}_k)^{\mathrm{Br}}$ explicitly.

In the next proposition we show that evaluation maps are continuous. In the proof we use the following constructions.

To every central simple algebra $A$ over a field $K$, there is associated a variety $V_A$ over $K$ called a *Severi–Brauer variety*. This is a twisted form of projective space: if $A$ has dimension $n^2$ over $K$, then, over a separable closure of $K$, the variety $V_A$ becomes isomorphic to $\mathbf{P}^{n-1}$. For details of the correspondence between central simple algebras and Severi–Brauer varieties, see Gille and Szamuely (2006, Chapter 5). For us, the most important property of the Severi–Brauer variety $V_A$ is that $V_A(K)$ is non-empty if and only if $A$ is isomorphic to a matrix algebra over $K$, that is, $A$ represents the trivial class in $\mathrm{Br}\,K$.

Let $Y$ be a smooth variety over a field $K$, and $\alpha$ a class in $\mathrm{Br}\,Y$. We can think of $\alpha$ as a family of elements of $\mathrm{Br}\,K$, parametrised by the points of $Y$; so it is not too much of a leap to think of the corresponding family of Severi–Brauer varieties parametrised by the points of $Y$. This idea turns out to be accurate, as described by Grothendieck (1968, Section I.8): it is possible to construct a *relative* Severi–Brauer variety $T_\alpha$ over $K$, equipped with a smooth morphism $T_\alpha \to Y$, such that the fibre $(T_\alpha)_P$ at a point $P \in Y(K)$ is a Severi–Brauer variety representing the Brauer class $\alpha(P)$. In particular, the fibre $(T_\alpha)_P$ has a $K$-point if and only if $\alpha(P)$ is trivial.

We now analyse the local situation.

**Proposition 13.1.9** *Let $Y$ be a smooth variety over a local field $K$, and let $\alpha$ be a class in* $\mathrm{Br}\,Y$*. The evaluation map $\alpha\colon Y(K) \to \mathrm{Br}\,K$ is locally constant for the analytic topology on $Y(K)$.*

*Proof* We give a proof of this fact using the constructions sketched above.

We first show that, supposing $\alpha(P) = 0$ at a point $P \in Y(K)$, there is some neighbourhood $U$ of $P$ in the analytic topology on $Y(K)$ such that $\alpha(Q) = 0$ holds for all $Q \in U$. To see this, let $T_\alpha \to Y$ be the relative Severi–Brauer variety associated to $\alpha$. By assumption, the class $\alpha(P)$ is trivial in $\mathrm{Br}\,K$; therefore the fibre $(T_\alpha)_P$ contains a $K$-point. Moreover, the morphism $f\colon T_\alpha \to Y$ is a smooth morphism of algebraic varieties, and hence the function $f_K\colon T_\alpha(K) \to Y(K)$ is a submersion of analytic manifolds over $K$. Thus, the Inverse Function Theorem (see Serre, 2006, Section II.III.10.2) shows that there is an analytic neighbourhood $U$ of $P$ over which $f_K$ admits an analytic section; in particular, the fibres $(T_\alpha)_Q$, for $Q \in U$, all contain a $K$-point. Hence $\alpha(Q)$ is trivial for all $Q \in U$.

To conclude, let us suppose, instead of $\alpha(P) = 0$, that we have $\alpha(P) = \beta$ for some $\beta \in \mathrm{Br}\,K$. Considering $\beta$ as an element of $\mathrm{Br}\,Y$ and applying the above argument to $\alpha - \beta$, we see that there is an analytic neighbourhood $U$ of $P$ in $Y(K)$ such that $\alpha(Q) = \beta$ holds for all $Q \in U$. In other words, the evaluation map $Y(K) \to \mathrm{Br}\,K$ is locally constant, proving the proposition. □

We return to the situation where $X$ is a smooth, projective, geometrically irreducible variety over a number field $k$.

**Corollary 13.1.10** *For each class $\alpha$ in* $\operatorname{Br} X$*, the adelic evaluation map* (13.2) *is continuous.*

*Proof* By Proposition 13.1.1, there is a finite set $S$ of places of $k$ such that the evaluation map $X(\mathbf{A}_k) \to \mathbf{Q}/\mathbf{Z}$ factors through the projection $\pi_S \colon X(\mathbf{A}_k) \to \prod_{v \in S} X(k_v)$. By definition of the adelic topology, the projection $\pi_S$ is continuous and using Proposition 13.1.9 we conclude that the evaluation map is continuous. □

**Corollary 13.1.11** *For each subset $B$ of* $\operatorname{Br} X$*, the set $X(\mathbf{A}_k)^B$ is closed in $X(\mathbf{A}_k)$.*

*Proof* For each $\alpha \in B$, the set $X(\mathbf{A}_k)^\alpha$ is the inverse image of 0 under the adelic evaluation map, so is closed. Therefore $X(\mathbf{A}_k)^B$ is an intersection of closed sets, so is also closed. □

The reason that the definition of the Brauer–Manin obstruction is so useful is that the sets $X(\mathbf{A}_k)^\alpha$ are often explicitly computable; for certain classes of varieties, we can even compute the set $X(\mathbf{A}_k)^{\mathrm{Br}}$ effectively. We look at this problem in more detail in Section 15.4.

## 13.2 Examples

**Example 13.2.1** Let $X$ be the non-singular del Pezzo surface of degree 4 of Example 2.3.5 defined by the equations

$$\begin{cases} uv = x^2 - 5y^2 \\ (u+v)(u+2v) = x^2 - 5z^2 \end{cases}$$

and let $\mathscr{A}$ be the quaternion algebra

$$\mathscr{A} = \left(5, \frac{u}{u+v}\right)$$

over $\kappa(X)$. In Example 12.1.2, we saw that the class $\alpha \in \operatorname{Br} \kappa(X)$ of $\mathscr{A}$ lies in $\operatorname{Br} X$. Let us now show that there is a Brauer–Manin obstruction to the Hasse principle on $X$ coming from $\alpha$.

We will describe the map $X(\mathbf{Q}_v) \to \mathbf{Q}/\mathbf{Z}$, given by $P \mapsto \operatorname{inv}_v \alpha(P)$, separately for each place $v$. As noted in Example 12.1.7, it is often enough to evaluate the functions in the definition of $\mathscr{A}$ to compute the evaluation $\alpha(P)$.

At the real place $\infty$, notice that 5 is positive and hence a square in $\mathbf{R}$. Let $P = [u_0, v_0, x_0, y_0, z_0]$ in $X(\mathbf{R})$ be a point such that $u_0$ and $v_0$ are non-zero. Then $\alpha(P)$ is the class of the trivial algebra $(5, u_0/(u_0+v_0))_{\mathbf{R}}$ and

$\mathrm{inv}_\infty\,\alpha(P) = 0$. Since the map $P \mapsto \mathrm{inv}_\infty\,\alpha(P)$ is locally constant on $X(\mathbf{R})$ by Proposition 13.1.9, it follows that it is zero everywhere.

At a place corresponding to an odd prime $p$ such that 5 is a square in $\mathbf{Q}_p$, the same argument works and shows that $\mathrm{inv}_p\,\alpha(P) = 0$ for all $P \in X(\mathbf{Q}_p)$.

Now suppose that $p \neq 5$ is an odd prime, such that 5 is not a square in $\mathbf{Q}_p$ and therefore not a square in $\mathbf{F}_p$ and let $P$ be a point in $X(\mathbf{Q}_p)$. Choose coordinates $P = [u_0, v_0, x_0, y_0, z_0]$ such that $u_0, v_0, x_0, y_0, z_0$ all lie in $\mathbf{Z}_p$ and are not all divisible by $p$. Reducing the equations modulo $p$, we deduce that $u_0$ and $v_0$ can never be both divisible by $p$, since otherwise $x_0/y_0$ and $x_0/z_0$ would be square roots of 5 in $\mathbf{F}_p$. Similarly, $p$ cannot divide both $u_0 + v_0$ and $u_0 + 2v_0$. It follows that, for each $P = [u, v, x, y, z] \in X(\mathbf{Q}_p)$, at least one of the expressions

$$\left(5, \frac{u}{u+v}\right), \quad \left(5, \frac{v}{u+v}\right), \quad \left(5, \frac{u}{u+2v}\right), \quad \left(5, \frac{v}{u+2v}\right)$$

is of the form $(5, b)$ with $b \in \mathbf{Z}_p^\times$, and therefore defines a quaternion algebra over $\mathbf{Q}_p$ whose class is $\alpha(P)$. By Proposition 10.1.6 the class $\alpha(P)$ is trivial in $\mathrm{Br}\,\mathbf{Q}_p$. We conclude that $\mathrm{inv}_p\,\alpha(P) = 0$ for all $P$ in $X(\mathbf{Q}_p)$.

Next, we analyse the prime 2. Solving the equations defining $X$ modulo 8 we find that $u$ and $v$ cannot both be even at a point of $X(\mathbf{Q}_2)$. (Note that working modulo 2 is not enough, since $(0, 0, 1, 1, 1) \in \mathbf{F}_2^5$ is a solution to the equations.) As before, for each $P \in X(\mathbf{Q}_2)$, one of $u, v$ is odd, and similarly one of $(u + v), (u + 2v)$ is odd. The formula of Proposition 10.1.6 shows that the Hilbert symbol $(5, b)_2$ is 1 whenever $b \in \mathbf{Z}_2$ is odd, so once again we conclude that $\mathrm{inv}_2\,\alpha(P) = 0$ for all $P \in X(\mathbf{Q}_2)$.

Finally, we look at the prime 5. Let $P$ be a point in $X(\mathbf{Q}_5)$. Choose coordinates $P = [u_0, v_0, x_0, y_0, z_0]$ such that $u_0, v_0, x_0, y_0, z_0$ all lie in $\mathbf{Z}_5$ and are not all divisible by 5. Reducing the coordinates of $P$ modulo 5, we obtain a point $\bar{P}$ in $\mathbf{P}^4_{\mathbf{F}_5}$ satisfying the equations defining $X$. Let $X_5$ be the variety over $\mathbf{F}_5$ defined by the reduction modulo 5 of the equations defining $X$. The variety $X_5$ is the union of four planes, meeting in a common line; two of these planes are defined over $\mathbf{F}_5$ and the other two are quadratic and conjugate. The two defined over $\mathbf{F}_5$, which therefore contain all the points of $X_5(\mathbf{F}_5)$, are $\{u = v = x\}$ and $\{u = v = -x\}$. The line of intersection of these planes is $\{u = v = x = 0\}$, but no point of $X(\mathbf{Q}_5)$ reduces to a point on this line, as can be seen easily reducing the equations modulo 25. Therefore the point $P$ in $X(\mathbf{Q}_5)$ satisfies $u_0 \equiv v_0 \equiv \pm x_0 \pmod 5$ with $u_0, v_0, x_0$ all being units in $\mathbf{Z}_5$. This means that $b = u_0/(u_0 + v_0)$ is congruent to 3 modulo 5, and the formula of Proposition 10.1.6 gives $(5, b)_5 = -1$. We deduce that $\mathrm{inv}_5\,\alpha(P) = \frac{1}{2}$ for all $P \in X(\mathbf{Q}_5)$.

To summarise, let $v$ be a place of $\mathbf{Q}$; we have proved that $\mathrm{inv}_v\,\alpha(P) = 0$

for all $P \in X(\mathbf{Q}_v)$ where $v \neq 5$, and that $\mathrm{inv}_5\,\alpha(P) = \frac{1}{2}$ for all $P \in X(\mathbf{Q}_5)$. It follows that

$$\sum_v \mathrm{inv}_v\,\alpha(P_v) = \tfrac{1}{2} \quad \text{for all } (P_v) \in X(\mathbf{A}_\mathbf{Q}).$$

So $X(\mathbf{A}_\mathbf{Q})^\alpha$ is empty, and therefore there is a Brauer–Manin obstruction to the Hasse principle on $X$.

We conclude by showing that the Brauer–Manin obstruction also explains the phenomenon of Example 2.3.8.

**Example 13.2.2** Let $S$ be the singular cubic surface of Example (2.3.8), defined in $\mathbf{P}^3_\mathbf{Q}$ by the equation

$$T(X^2+Y^2) = (4Z-7T)(Z^2-2T^2).$$

The two singular points of $S$ are $(X:Y:Z:T) = (\pm i:1:0:0)$ where $i^2 = -1$. Let $U$ denote the complement in $S$ of these two points. Any rational point of $S$ must be contained in $U$, since neither of the singular points is rational.

Let $\mathscr{A}$ be the quaternion algebra over $\kappa(S)$ defined by

$$\mathscr{A} = \left(-1, \frac{4Z-7T}{T}\right).$$

Let us ignore for the moment that we have not defined $\mathrm{Br}\,S$. We will show that the class of $\mathscr{A}$ gives a Brauer–Manin obstruction to weak approximation that explains why one connected component of $S(\mathbf{R})$ contains no rational points. In doing so, we will show that the class of $\mathscr{A}$ lies in $\mathrm{Br}\,U$. In fact, our calculations are sufficient to show that the class of $\mathscr{A}$ lies in $\mathrm{Br}\,\tilde{S}$ where $\tilde{S}$ is any smooth projective variety birational to $S$.

As before, we start by finding alternative ways of writing the quaternion algebra $\mathscr{A}$. Firstly, looking at the defining equations of $S$ we can see equalities

$$T(X^2+Y^2) + 2T^2(4Z-7T) = Z^2(4Z-7T)$$

and so

$$\frac{4Z-7T}{T} = \frac{X^2+Y^2+8ZT-14T^2}{Z^2}$$

as functions on $S$, immediately giving a new way of writing $\mathscr{A}$. Furthermore, since the denominator $Z^2$ is a square, we can replace it with any other square such as $X^2$, $Y^2$ or $T^2$ to get new quaternion algebras isomorphic to $\mathscr{A}$. Also, the defining equations give

$$\left(\frac{4Z-7T}{T}\right)\left(\frac{Z^2-2T^2}{T^2}\right) = \frac{X^2+Y^2}{T^2} = \mathrm{N}_{\kappa(S)(i)/\kappa(S)}\left(\frac{X+iY}{T}\right)$$

and so the algebra $(-1,(Z^2-2T^2)/T^2)$ is isomorphic to $\mathscr{A}$, and again the denominator here may be replaced by any square. In this way we find a set of isomorphic quaternion algebras over $\kappa(S)$:

$$\left(-1,\frac{4Z-7T}{T}\right),\quad \left(-1,\frac{X^2+Y^2+8ZT-14T^2}{\text{(square)}}\right),\quad \left(-1,\frac{Z^2-2T^2}{\text{(square)}}\right).$$

Let $\alpha$ be the common class in $\operatorname{Br}\kappa(S)$ of these algebras. At any point $P$ of $U$, at least one of these expressions defines an Azumaya algebra over the local ring $\mathscr{O}_{S,P}$ and so can be used to evaluate the class $\alpha$ at $P$. Indeed, if $Z^2-2T^2$ vanishes at a point, then neither $4Z-7T$ nor $T$ vanishes unless we have $Z=T=0$. In that case, the middle expression above defines an Azumaya algebra unless we also have $X^2+Y^2=0$, but the only points where that happens are the singular points, which are not in $U$.

We now analyse the invariant of $\alpha$ at each place separately. Let $p$ be prime. If $p\equiv 1 \pmod 4$ then $-1$ is a square in $\mathbf{Q}_p$, and so $\operatorname{inv}_p\alpha(P)$ is 0 for all $P\in U(\mathbf{Q}_p)$.

If $p\equiv 3 \pmod 4$ then let $P$ be a point of $U(\mathbf{Q}_p)$ and choose coordinates $P=[x,y,z,t]$ such that $x,y,z,t$ lie in $\mathbf{Z}_p$ and are not all divisible by $p$. We argue as in the previous example: we prove that the local invariant is always zero by showing that at least one of $(4z-7t)/t, x^2+y^2+8zt-14t^2, z^2-2t^2$ is a $p$-adic unit. First we show that, if $t$ is divisible by $p$, then $x^2+y^2+8zt-14t^2$ is a $p$-adic unit. Indeed, suppose that $t$ is divisible by $p$; the equation of $S$ shows that $z$ is also divisible by $p$. Since $p\equiv 3 \pmod 4$, the value $x^2+y^2+8zt-14t^2$ could be divisible by $p$ only if both $x$ and $y$ were divisible by $p$, contrary to our assumptions. We reduce to the case in which $t$ is not divisible by $p$. Suppose that $p$ divides both $4z-7t$ and $z^2-2t^2$; then $p$ divides $16(z^2-2t^2)-(4z+7t)(4z-7t)=17t^2$. Since $p$ does not divide $t$, it follows that $p$ divides 17, but 17 is not congruent to 3 modulo 4. We conclude that $p$ does not divides at most one of $4z-7t$ and $z^2-2t^2$, as we wanted to show.

We analyse what happens at 2. From the formula in Proposition 10.1.6(iii), we see that the Hilbert symbol $(-1,2^\beta v)_2$ takes the value 1 if $v\equiv 1 \pmod 4$, and takes the value $-1$ if $v\equiv 3 \pmod 4$. Let $P$ be a point of $U(\mathbf{Q}_2)$ and choose coordinates $P=[x,y,z,t]$ such that $x,y,z,t$ lie in $\mathbf{Z}_2$ and are not all even. If $t$ is odd, then we have $(4z-7t)/t\equiv 1 \pmod 4$ and therefore $\alpha(P)$ is trivial. If $t$ is even and $z$ is odd, then we have $z^2-2t^2\equiv 1 \pmod 4$ and again $\alpha(P)$ is trivial. Lastly, if $t$ and $z$ are both even, then $x,y$ are not both even, and therefore the value $x^2+y^2+8zt-14t^2$ is congruent to either 1 or 2 modulo 8, so that $\alpha(P)$ is trivial.

Finally, over $\mathbf{R}$, we see that $(4Z-7T)/T$ is non-negative on one component of $S(\mathbf{R})$, and strictly positive on a dense open subset of that component, so $\mathrm{inv}_\infty \alpha = 0$ on that component. On the other component, however, $(4Z-7T)/T$ is strictly negative, and therefore $\mathrm{inv}_\infty \alpha = \frac{1}{2}$ on that component. We deduce that rational points can only be found in the component where $Z/T \geq \frac{7}{4}$, giving a Brauer–Manin obstruction to weak approximation on $S$.

In the example above, we worked with the open variety $U$ obtained by removing the singular locus of a cubic surface $S$. It is a general fact that for every projective surface $X$ there is a smooth projective surface $X'$ and a morphism $X' \to X$ that is an isomorphism on the smooth locus of $X$. As a consequence, every smooth surface can be embedded as an open subset of a smooth projective surface. Note that, if the surface $X$ is not smooth, then the smooth surface $X'$ is not uniquely determined by the stated conditions, since it is always possible to replace $X'$ by the blow-up at some point. Of course, all these surfaces $X'$ are birational to one another.

Going back to the surface $S$ of Example 13.2.2, we worked on the open subset $U$, since we have not defined Brauer groups of singular varieties, and so we cannot talk about the Brauer group of $S$. At the same time, we were able to find an element $\alpha$ of $\mathrm{Br}\,U$ giving an obstruction to weak approximation. By an explicit computation, we discovered that the invariant map for the class $\alpha$ is trivial at almost all places: there is no reason to expect this on a non-projective variety. In fact, if $S'$ is any smooth projective surface containing $U$, then the class $\alpha$ considered as a class in $\mathrm{Br}\,\kappa(S')$ does indeed lie in $\mathrm{Br}\,S'$. We can see this in different ways: we could explicitly construct an $S'$ and check that the class $\alpha$ is unramified on $S'$; we could also use the result of Harari (1994) mentioned in Remark 13.1.3, since the local invariant maps of $\alpha$ are almost all zero. We can also formulate this conclusion as stating that the class $\alpha$ lies in the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(\kappa(U)/k)$ (see Remark 12.2.10).

**Exercise 13.2.3** Find a smooth projective surface $S'$ (Exercise **??**) containing $U$ and show explicitly that the class $\alpha$ lies in the Brauer group of $S'$.

# 15

# The Brauer group and cohomology

In this chapter we look at how the techniques of Galois cohomology can be used to find explicit elements of the Brauer group of a variety.

We begin by relating the Brauer group to some Galois cohomology groups. In particular, we define a *residue map* which allows us to describe the Brauer group of a discrete valuation ring as a subgroup of the Brauer group of its field of fractions. Using the residue map, we obtain a complete description of the Brauer group of a local field. We also use the residue maps to give a description, in terms of Galois cohomology, of the *algebraic* part of the Brauer group of a variety. For more information on the transcendental part of the Brauer group, it would be necessary to use the more general techniques of étale cohomology, which will not be discussed in this book.

## 15.1 Residue maps

Let $X$ be a smooth, irreducible variety over a field $k$ of characteristic 0. The Brauer group of $X$ is the subgroup of the Brauer group of the function field $X$ consisting of those elements which are unramified at all points of $X$; as a consequence of the Purity Theorem we saw, in Theorem 12.2.8, that $\mathrm{Br}X$ is also the subgroup of $\mathrm{Br}\,\kappa(X)$ consisting of those elements unramified at all *divisors* on $X$. Let us now give a cohomological description of what it means for an element of $\mathrm{Br}\,\kappa(X)$ to be unramified at a divisor.

### 15.1.1 The group $\mathrm{H}^2(G, \mathbf{Z})$

For any finite group $G$, the second cohomology group $\mathrm{H}^2(G, \mathbf{Z})$ is naturally isomorphic to the character group $\mathrm{Hom}(G, \mathbf{Q}/\mathbf{Z})$; in this section, we will prove this, and explicitly describe the isomorphism in terms of cocycles.

Fix a finite group $G$. Consider the exact sequence of $G$-modules with trivial action

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0. \tag{15.1}$$

For any positive integer $n$, the multiplication-by-$n$ map $\mathbf{Q} \xrightarrow{\times n} \mathbf{Q}$ is an isomorphism. It follows that, for any finite group $G$ of order dividing $n$ acting trivially on $\mathbf{Q}$, the induced map on cohomology groups

$$\mathrm{H}^i(G,\mathbf{Q}) \xrightarrow{\times n} \mathrm{H}^i(G,\mathbf{Q})$$

is an isomorphism for any $i > 0$; combining this with Proposition 14.4.10, we deduce that it is also the zero map, and so $\mathrm{H}^i(G,\mathbf{Q}) = 0$ for all $i > 0$. It therefore follows from the exact sequence (15.1) that there are isomorphisms $\delta_i \colon \mathrm{H}^i(G,\mathbf{Q}/\mathbf{Z}) \to \mathrm{H}^{i+1}(G,\mathbf{Z})$ for all $i \geq 1$. Taking $i = 1$, we obtain the following proposition.

**Proposition 15.1.1** *Let $G$ be a finite group, acting trivially on the $G$-module $\mathbf{Z}$. Then there is an isomorphism*

$$\mathrm{Hom}(G,\mathbf{Q}/\mathbf{Z}) \cong \mathrm{H}^2(G,\mathbf{Z}).$$

*Proof* This follows from the above discussion, together with the fact, proved in Exercise 14.2.5, that $\mathrm{H}^1(G,\mathbf{Q}/\mathbf{Z})$ can be identified with $\mathrm{Hom}(G,\mathbf{Q}/\mathbf{Z})$. □

Let us give an explicit description of this isomorphism. Fix a finite group $G$ and a homomorphism $\alpha \colon G \to \mathbf{Q}/\mathbf{Z}$, which we consider as a 1-cocycle. By the description (14.2) of the connecting homomorphism $\delta \colon \mathrm{H}^1(G,\mathbf{Q}/\mathbf{Z}) \to \mathrm{H}^2(G,\mathbf{Z})$, we compute $\delta\alpha$ by first lifting $\alpha$ to a function $\hat{\alpha} \colon G \to \mathbf{Q}$, which is in general not a homomorphism, and then setting

$$\delta\alpha(g_1,g_2) = g_1\hat{\alpha}(g_2) + \hat{\alpha}(g_1) - \hat{\alpha}(g_1g_2).$$

Since $\alpha$ was a homomorphism, it is easy to see that $\delta\alpha$ takes values in $\mathbf{Z}$. As the action of $G$ is trivial, the equation above simplifies to

$$\delta\alpha(g_1,g_2) = \hat{\alpha}(g_2) + \hat{\alpha}(g_1) - \hat{\alpha}(g_1g_2). \tag{15.2}$$

As a particular case of this calculation, we deduce the following result.

**Proposition 15.1.2** *Let $G$ be a cyclic group of order $n$, and fix a generator $g$ of $G$. Then the function $\phi \colon G^2 \to \mathbf{Z}$ defined by*

$$\phi(g^i,g^j) = \begin{cases} 0 & \text{if } i+j < n; \\ 1 & \text{if } i+j \geq n; \end{cases}$$

*is a 2-cocycle, the class of which generates $\mathrm{H}^2(G,\mathbf{Z})$.*

*Proof* The group $\mathrm{Hom}(G,\mathbf{Q}/\mathbf{Z})$ is cyclic, generated by the homomorphism $\alpha\colon G \to \mathbf{Q}/\mathbf{Z}$ defined by $\alpha(g) = 1/n$. It follows from Proposition 15.1.1 that $\mathrm{H}^2(G,\mathbf{Z})$ is also cyclic of order $n$; all that remains is to prove that the class of $\phi$ is a generator. A lift of $\alpha$ to $\mathbf{Q}$ is the map $\hat{\alpha}\colon G \to \mathbf{Q}$ such that $\hat{\alpha}(g^i) = i/n$ when $0 \le i < n$. Applying the formula (15.2) and setting $\phi := \delta\alpha$ then gives the result. □

For now, let $R$ be a *complete* discrete valuation ring with field of fractions $K$ and residue field $k$, which we suppose to be perfect. (In particular, this is true if $k$ is finite, or if $k$ has characteristic zero.) In this context, we can define a *residue map*, which is a homomorphism from $\mathrm{Br}\,K$ to $\mathrm{H}^1(k,\mathbf{Q}/\mathbf{Z})$. Later, we will apply this to the discrete valuation associated to a divisor $D$ on a smooth variety $X$. The residue map at $D$ carries information about how an algebra over the function field of $X$ ramifies along $D$; in particular, we will see that the unramified algebras, those in the Brauer group of $X$, are precisely those having trivial residue along every divisor. We follow the treatment given by Colliot-Thélène and Swinnerton-Dyer (1994).

**Proposition 15.1.3** *Let $R$ be a complete discrete valuation ring with field of fractions $K$ and perfect residue field $k$. Let $A$ be a central simple algebra over $K$. Then there is a finite, unramified extension of $K$ which splits $A$.*

*Proof* Several different proofs are available in the literature. A direct proof is given by Serre (1968, XII.2) of the following more specific fact: that $A$ contains a maximal subfield which is unramified over $K$. □

Note that, when $k$ is perfect, an unramified extension $L/K$ is necessarily separable: this can be seen by looking at the trace form on $L$.

*Remark* 15.1.4 In Lemma 3.4 of Auslander and Brumer (1968), it is shown that Proposition 15.1.3 holds without the assumption that $R$ is complete, though in that case the unramified splitting field is not necessarily contained in $A$.

Let $A$ be a central simple algebra over $K$. By Proposition 15.1.3, there is a finite, unramified extension $L/K$ over which $A$ splits. Taking the normal closure, we may assume that $L/K$ is Galois, so that the class of $A$ defines a cohomology class in $\mathrm{H}^2(L/K,L^\times) \subset \mathrm{H}^2(K,\bar{K}^\times)$. Let $v_L\colon L^\times \to \mathbf{Z}$ be the valuation on $L$; as this map respects the Galois action, there is an induced map on cohomology

$$\mathrm{H}^2(L/K,L^\times) \longrightarrow \mathrm{H}^2(L/K,\mathbf{Z}).$$

Next, Proposition 15.1.1 gives an isomorphism

$$\delta\colon \mathrm{Hom}(\mathrm{Gal}(L/K),\mathbf{Q}/\mathbf{Z}) = \mathrm{H}^1(L/K,\mathbf{Q}/\mathbf{Z}) \to \mathrm{H}^2(L/K,\mathbf{Z}).$$

Finally, since $L/K$ is unramified, there is a natural isomorphism of Galois groups $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(\ell/k)$, where $\ell$ is the residue field of $L$. Putting these various maps together as follows, we obtain the residue map.

**Definition 15.1.5** Let $R$ be a complete discrete valuation ring with field of fractions $K$ and perfect residue field $k$, and let $L$ be a finite, unramified, Galois extension of $K$ with residue field $\ell$. The *residue map* $\partial_{R,L}\colon \mathrm{Br}(L/K) \to \mathrm{H}^1(\ell/k, \mathbf{Q}/\mathbf{Z})$ is the composition

$$\mathrm{Br}(L/K) \cong \mathrm{H}^2(L/K, L^\times) \xrightarrow{v_L} \mathrm{H}^2(L/K, \mathbf{Z}) \xrightarrow{\delta^{-1}} \mathrm{H}^1(L/K, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \mathrm{H}^1(\ell/k, \mathbf{Q}/\mathbf{Z})$$

where $v_L$ and $\delta$ are as above.

**Exercise 15.1.6** Let $n$ denote the degree $[L:K]$, and let $\pi$ be a uniformising element in $K$. Show that the homomorphism $\mathrm{H}^1(\ell/k, \mathbf{Q}/\mathbf{Z}) \to \mathrm{H}^2(L/K, L^\times)$ defined by $\chi \mapsto \pi \cup \delta\chi$ is a right inverse to $\partial_{R,L}$.

If $L'$ is another finite, unramified, Galois extension of $K$ containing $L$, with residue field $\ell'$, then $v_{L'}$ restricts to $v_L$ on $L$, and it follows that the diagram

$$\begin{array}{ccc} \mathrm{H}^2(L'/K, (L')^\times) & \xrightarrow{\partial_{R,L'}} & \mathrm{H}^1(\ell'/k, \mathbf{Q}/\mathbf{Z}) \\ \mathrm{Infl}\uparrow & & \mathrm{Infl}\uparrow \\ \mathrm{H}^2(L/K, L^\times) & \xrightarrow{\partial_{R,L}} & \mathrm{H}^1(\ell/k, \mathbf{Q}/\mathbf{Z}) \end{array}$$

commutes. As $L$ ranges through all finite, unramified extensions of $K$, so $\ell$ ranges through all finite extensions of $k$; taking the limit, the residue maps $\partial_{R,L}$ therefore induce a map from $\mathrm{Br}\,K$ to $\mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z})$.

**Definition 15.1.7** Let $R$ be a complete discrete valuation ring, with field of fractions $K$ and perfect residue field $k$. The *residue map* associated to $R$ is the group homomorphism

$$\partial_R\colon \mathrm{Br}\,K \longrightarrow \mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z})$$

which is the limit of the residue maps $\partial_{R,L}$ as $L$ varies over all finite, unramified extensions of $K$.

More explicitly, suppose that $R$ and $K$ are as in Definition 15.1.7, and that we are given a central simple algebra $A$ over $K$. By Proposition 15.1.3, there is a finite, unramified extension of $K$ that splits $A$. Taking the normal closure, let $L/K$ be a finite, unramified, Galois extension splitting $A$, and let $\ell$ be the residue field of $L$; then $\partial_R(A)$ is the image of $\partial_{R,L}(A)$ under the (injective) inflation homomorphism $\mathrm{H}^1(\ell/k, \mathbf{Q}/\mathbf{Z}) \to \mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z})$.

**Example 15.1.8** Take $R$ to be $\mathbf{Q}[[t]]$, the ring of formal power series in the variable $t$ with coefficients in $\mathbf{Q}$, so that $K$ is $\mathbf{Q}((t))$, the corresponding field of formal Laurent series. Consider the quaternion algebra $A = (2,t)_K$. There are two obvious field extensions of $K$ contained in $A$: the field $L = K(\sqrt{2}) = \mathbf{Q}(\sqrt{2})((t))$; and the field $L' = K(\sqrt{t}) = \mathbf{Q}((s))$, where $s^2 = t$. (There are also many other quadratic extensions of $K$ contained in $A$, but these two are sufficient for this example.) Of these field extensions, the second is ramified: a uniformising element in $L'$ is $s$, and by the relation $t = s^2$ we see that $v_{L'}(t) = 2$. On the other hand, $L$ is unramified over $K$, as $t$ is a uniformising element in $L$ and so $v_L(t) = 1$. We may therefore use the extension $L/K$ to compute the residue of $A$.

Let $G$ be the Galois group of $L/K$, which is cyclic of order 2, and let $g$ be the generator of $G$. By Proposition **??**, the class in $\mathrm{H}^2(G,L^\times)$ corresponding to $A$ is represented by the cocycle $\psi\colon G^2 \to L^\times$ defined by

$$\psi(\mathrm{id},\mathrm{id}) = \psi(\mathrm{id},g) = \psi(g,\mathrm{id}) = 1, \quad \psi(g,g) = t.$$

Applying $v_L$ to this cocycle gives a 2-cocycle with values in $\mathbf{Z}$:

$$\phi(\mathrm{id},\mathrm{id}) = \phi(\mathrm{id},g) = \phi(g,\mathrm{id}) = 0, \quad \phi(g,g) = 1.$$

According to Proposition 15.1.2, this corresponds under the isomorphism $\delta$ to the homomorphism $\alpha\colon G \to \mathbf{Q}/\mathbf{Z}$ defined by $\alpha(g) = \frac{1}{2}$. Now the inflation map $\mathrm{Infl}\colon \mathrm{H}^1(L/K,\mathbf{Q}/\mathbf{Z}) \to \mathrm{H}^1(K,\mathbf{Q}/\mathbf{Z})$ is simply pulling back homomorphisms $G \to \mathbf{Q}/\mathbf{Z}$ to homomorphisms $\mathrm{Gal}(\bar{K}/K) \to \mathbf{Q}/\mathbf{Z}$. We finally obtain $\partial_R(A) = \tilde{\alpha} \in \mathrm{H}^1(K,\mathbf{Q}/\mathbf{Z})$, where

$$\tilde{\alpha}(\sigma) = \begin{cases} 0 & \text{if } \sigma(s) = s; \\ \frac{1}{2} & \text{if } \sigma(s) = -s. \end{cases}$$

We may also define the residue map when $R$ is not complete, as follows.

**Definition 15.1.9** Let $R$ be any discrete valuation ring, with field of fractions $K$ and perfect residue field $k$. Denote by $\widehat{R}$ the completion of $R$, and by $\widehat{K}$ the completion of $K$. The *residue map* associated to $R$ is the composite

$$\mathrm{Br}\,K \to \mathrm{Br}\,\widehat{K} \xrightarrow{\partial_{\widehat{R}}} \mathrm{H}^1(k,\mathbf{Q}/\mathbf{Z}).$$

The importance of the residue map is in the following statement, which says that the classes in $\mathrm{Br}\,K$ coming from Azumaya algebras over $R$ are precisely those with zero residue.

**Theorem 15.1.10** *Let $R$ be a discrete valuation ring with field of fractions $K$*

*and perfect residue field k. Then there is a short exact sequence*

$$0 \to \operatorname{Br} R \to \operatorname{Br} K \xrightarrow{\partial_R} \mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z}) \to 0.$$

*Proof* This is Theorem 3.3 of Auslander and Brumer (1968), and we sketch their proof. Firstly, every central simple algebra over $K$ is split by a finite, unramified Galois extension, so it is enough to prove the corresponding statement for finite extensions. Let $L/K$ be an unramified extension and denote by $U(L)$ the group of units in the valuation ring of $L$. By definition there is a short exact sequence

$$0 \to U(L) \to L^\times \xrightarrow{v_L} \mathbf{Z} \to 0 \tag{15.3}$$

that is split by sending $1 \in \mathbf{Z}$ to a uniformiser in $K^\times \subset L^\times$. It is shown by Auslander and Goldman (1960, Theorem A.15) that the subgroup of $\operatorname{Br} R$ split by $L$ can be identified with the cohomology group $\mathrm{H}^2(L/K, U(L))$. The result now follows from the long exact cohomology sequence associated to (15.3). □

When combined with Theorem 12.2.8, this result allows us to state a cohomological version of the Purity Theorem.

**Theorem 15.1.11** (Purity Theorem, second form) *Let $X$ be a smooth, irreducible variety over a field of characteristic zero. For each prime divisor $Z$ on $X$, let $\partial_Z$ denote the residue map $\partial_{\mathscr{O}_{X,Z}} \colon \operatorname{Br} \kappa(X) \to \mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z})$ associated to the discrete valuation ring $\mathscr{O}_{X,Z}$. Then there is an exact sequence*

$$0 \to \operatorname{Br} X \to \operatorname{Br} \kappa(X) \xrightarrow{\oplus_Z \partial_Z} \bigoplus_Z \mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z})$$

*where the sum is taken over all prime divisors $Z$ on $X$.*

*Proof* The main part of the theorem, the exactness, comes directly from combining Theorem 12.2.8 with Theorem 15.1.10. But we must first show that any class in $\operatorname{Br} \kappa(X)$ has non-zero residue at only finitely many prime divisors $Z$.

Let $\alpha$ be a class in $\operatorname{Br} \kappa(X)$. By Corollary 12.2.6, there is a dense affine open subset $U \subset X$ such that $\alpha$ lies in $\operatorname{Br} U$. Now let $Z$ be any prime divisor on $X$ meeting $U$; then $\alpha$ is unramified at $Z$, and so Theorem 15.1.10 gives $\partial_Z(\alpha) = 0$. Thus $\alpha$ can have non-zero residue only at prime divisors not meeting $U$, of which there are finitely many. So the image of $\oplus_Z \partial_Z$ does indeed lie in the direct sum of the $\mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z})$. Now Theorem 15.1.10 shows that any class in the kernel of $\oplus_Z \partial_Z$ lies in the intersection $\bigcap_Z \operatorname{Br} \mathscr{O}_{X,Z} \subset \operatorname{Br} \kappa(X)$, which by Theorem 12.2.8 is equal to $\operatorname{Br} X$. □

## 15.2 The Brauer group of a local field

The discussion of residue maps in the previous section, and Theorem 15.1.10, allow us to calculate the structure of the Brauer group of a local field.

**Theorem 15.2.1** *Let $p$ be prime, and let $K$ be a finite extension of $\mathbf{Q}_p$. Then there is a canonical isomorphism* $\operatorname{Br} K \to \mathbf{Q}/\mathbf{Z}$.

*Proof* Let $R$ denote the ring of integers of $K$, and $k$ the residue field, which is finite. Theorem 15.1.10 tells us that there is an exact sequence

$$0 \to \operatorname{Br} R \to \operatorname{Br} K \xrightarrow{\partial} \mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z}) \to 0.$$

Now $\operatorname{Br} R$ is trivial by Corollary **??**, and so $\partial$ is an isomorphism. To complete the proof, we must give a canonical isomorphism between $\mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z})$ and $\mathbf{Q}/\mathbf{Z}$. The key to this is that, for each finite extension $\ell/k$, there is a canonical generator for $\operatorname{Gal}(\ell/k)$, namely the Frobenius automorphism $F_\ell$. If $\ell/k$ is an extension of degree $n$, we define an injective homomorphism $\phi_\ell$ from $\mathrm{H}^1(\ell/k, \mathbf{Q}/\mathbf{Z}) = \operatorname{Hom}(\operatorname{Gal}(\ell/k), \mathbf{Q}/\mathbf{Z})$ to $\mathbf{Q}/\mathbf{Z}$ by defining $\phi_\ell(\alpha) = \alpha(F_\ell)$. It is straightforward to check that these homomorphisms fit together, in the sense that, for any extension $\ell'/\ell$, the map $\phi_{\ell'}$ restricts to $\phi_\ell$ on $\operatorname{Hom}(\operatorname{Gal}(\ell/k), \mathbf{Q}/\mathbf{Z}) \subseteq \operatorname{Hom}(\operatorname{Gal}(\ell'/k), \mathbf{Q}/\mathbf{Z})$. The various homomorphisms $\phi_\ell$ therefore define an injective homomorphism $\mathrm{H}^1(k, \mathbf{Q}/\mathbf{Z}) \to \mathbf{Q}/\mathbf{Z}$; it is surjective as well, since the image contains elements of every order. □

**Definition 15.2.2** Let $p$ be prime, and let $K$ be a finite extension of $\mathbf{Q}_p$. The canonical isomorphism of Theorem 15.2.1 is called the *invariant map* and is denoted

$$\operatorname{inv}_K \colon \operatorname{Br} K \longrightarrow \mathbf{Q}/\mathbf{Z}.$$

In order to extend this definition to all completions of a number field, we also define

$$\operatorname{inv}_{\mathbf{R}} \colon \operatorname{Br} \mathbf{R} \to \mathbf{Q}/\mathbf{Z} \quad \text{and} \quad \operatorname{inv}_{\mathbf{C}} \colon \operatorname{Br} \mathbf{C} \to \mathbf{Q}/\mathbf{Z}$$

to be the unique injective homomorphism in each case (sending $\operatorname{Br} \mathbf{R}$ to $\{0, 1/2\}$ and $\operatorname{Br} \mathbf{C}$ to $\{0\}$). Now let $k$ be a number field; for each place $v$ of $k$, we denote by $\operatorname{inv}_v$ the composition

$$\operatorname{inv}_v \colon \operatorname{Br} k \to \operatorname{Br} k_v \xrightarrow{\operatorname{inv}_{k_v}} \mathbf{Q}/\mathbf{Z}.$$

**Exercise 15.2.3** For any $x \in \mathbf{Q}/\mathbf{Z}$, find an explicit cyclic algebra $A$ over $\mathbf{Q}_p$ with $\operatorname{inv}_{\mathbf{Q}_p}(A) = x$. Deduce that every class in $\operatorname{Br} \mathbf{Q}_p$ is represented by a cyclic algebra.

## 15.3 The algebraic Brauer group

Throughout this section, let $X$ denote a smooth, geometrically irreducible variety over a field $k$. Recall from Section **??** that the *algebraic* part of the Brauer group of $X$, denoted $\mathrm{Br}_1 X$, consists of those classes in $\mathrm{Br}X$ which are split by a finite extension of the base field $k$. More precisely, a finite extension $\ell/k$ induces a finite extension of function fields $\kappa(X_\ell)/\kappa(X)$; a class $\alpha \in \mathrm{Br}X$ lies in $\mathrm{Br}_1 X$ if it is split by such an extension.

As $X$ is geometrically irreducible, the two fields $\ell$ and $\kappa(X)$ are linearly disjoint extensions of $k$. It follows that there is a canonical isomorphism

$$\mathrm{Gal}(\kappa(X_\ell)/\kappa(X)) \cong \mathrm{Gal}(\ell/k).$$

Using this isomorphism, we will write $\mathrm{H}^2(\ell/k, \kappa(X_\ell)^\times)$ as slightly less cumbersome notation for the subgroup

$$\mathrm{H}^2(\kappa(X_\ell)/\kappa(X), \kappa(X_\ell)^\times) \subset \mathrm{H}^2(\kappa(X), \overline{\kappa(X)}^\times) = \mathrm{Br}\,\kappa(X).$$

The definition of $\mathrm{Br}_1 X$ as consisting of those classes in $\mathrm{Br}\,\kappa(X)$ which are both unramified and split by a finite extension of $k$ means that, taking the limit over finite extensions $\ell/k$, we can identify

$$\mathrm{Br}_1 X = \mathrm{H}^2(k, \kappa(\bar{X})^\times) \cap \mathrm{Br}X,$$

the intersection taking place inside $\mathrm{Br}\,\kappa(X)$.

To arrive at a computationally useful description of $\mathrm{Br}_1 X$, we first re-interpret the residue map at a prime divisor of $X$. The following proposition can be thought of as saying that the natural homomorphism $\mathrm{H}^2(k, \kappa(\bar{X})^\times) \to \mathrm{H}^2(k, \mathrm{Div}\,\bar{X})$ "is" the restriction of all the residue maps to $\mathrm{Br}_1 X$.

**Proposition 15.3.1** *There is an injective homomorphism*

$$\phi \colon \mathrm{H}^2(k, \mathrm{Div}\bar{X}) \longrightarrow \bigoplus_Z \mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z}),$$

*where the sum is over all prime divisors $Z$ on $X$, such that the diagram*

$$\begin{array}{ccc} \mathrm{Br}\,\kappa(X) & \xrightarrow{\oplus_Z \partial_Z} & \bigoplus_Z \mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z}) \\ \uparrow & & \uparrow \phi \\ \mathrm{H}^2(k, \kappa(\bar{X})^\times) & \xrightarrow{\mathrm{div}} & \mathrm{H}^2(k, \mathrm{Div}\,\bar{X}) \end{array}$$

*commutes.*

*Proof* As described in Exercise 14.7.8, the Galois module $\mathrm{Div}\,\bar{X}$ splits as a direct sum of Galois modules $\mathrm{Div}_Z \bar{X}$, where $Z$ runs over the prime divisors on $X$. We will define injective homomorphisms

$$\phi_Z \colon \mathrm{H}^2(k, \mathrm{Div}_Z \bar{X}) \to \mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z})$$

and then take $\phi = \oplus_Z \phi_Z$.

Fix a prime divisor $Z$ on $X$. Over $\bar{k}$, $Z$ may decompose into several irreducible components $Z_i$; let $k'/k$ be the minimal field of definition of one component $Z_1$. By **??**, $k'$ can be taken to be the algebraic closure of $k$ within $\kappa(Z)$. It was shown in Exercise 14.7.8 that the Galois module $\mathrm{Div}_Z \bar{X}$ is isomorphic to the induced module $\mathrm{Ind}_{k'/k} \mathbf{Z}$, and therefore Shapiro's Lemma (Proposition 14.4.3) gives an isomorphism $\mathrm{H}^2(k, \mathrm{Div}_Z \bar{X}) \cong \mathrm{H}^2(k', \mathbf{Z})$. Let

$$j \colon \mathrm{Gal}(\overline{\kappa(Z)}/\kappa(Z)) \to \mathrm{Gal}(\bar{k}/k')$$

be the homomorphism defined by restricting an automorphism of $\overline{\kappa(Z)}$ to $\bar{k}$. We define $\phi_Z$ to be the composite

$$\mathrm{H}^2(k, \mathrm{Div}_Z \bar{X}) \cong \mathrm{H}^2(k', \mathbf{Z}) \cong \mathrm{H}^1(k', \mathbf{Q}/\mathbf{Z}) \xrightarrow{j^*} \mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z})$$

As $k'$ is algebraically closed in $\kappa(Z)$, the homomorphism $j$ is surjective; therefore $j^*$, and hence $\phi_Z$, is injective.

We must now show that the diagram does indeed commute. This follows in a messy but straightforward way from the definitions. Recall the definition of the residue map $\partial_Z$ as the composite homomorphism

$$\mathrm{Br}\, \kappa(X) \to \mathrm{Br}\, \widehat{\kappa(X)} \to \mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z})$$

where $\widehat{\kappa(X)}$ is the completion of $\kappa(X)$ with respect to the discrete valuation defined by $Z$. We are given a class in $\mathrm{Br}\, \kappa(X)$ which is split by the finite extension $\kappa(X_\ell)/\kappa(X)$; since this extension (like every extension coming from an extension of the base field) is unramified above $Z$, we can use it to compute the residue. (The point of passing to the completion when defining the residue map was to guarantee the existence of an unramified splitting field; here we are in the happy position of having an unramified splitting field even before passing to the completion.) For brevity, write $K = \kappa(X)$ and $L = \kappa(X_\ell)$. After possibly enlarging $\ell$, we may assume that $\ell/k$ is Galois, and that the divisors $Z_i$ are all defined over $\ell$ (and in particular that $k'$ is contained in $\ell$). Now the residue map on $\mathrm{H}^2(L/K, L^\times)$ was defined to be the composition

$$\mathrm{H}^2(L/K, L^\times) \xrightarrow{c} \mathrm{H}^2(\widehat{L}/\widehat{K}, \widehat{L}^\times) \xrightarrow{v_1} \mathrm{H}^2(\widehat{L}/\widehat{K}, \mathbf{Z}) \xrightarrow{\mathrm{Infl} \circ \delta^{-1}} \mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z}).$$

Here $\widehat{L}$ is the completion of $L$ under a chosen extension of the valuation $v_Z$; we will choose the valuation associated to the divisor $Z_1$ on $X_\ell$, which we denote by $v_1$. The map $c$ is as described in **??**. Now $\mathrm{Gal}(L/K)$ can be identified with $\mathrm{Gal}(\ell/k)$, and $\mathrm{Gal}(\widehat{L}/\widehat{K})$ can be identified with a subgroup, the decomposition group associated to the chosen valuation, which as we saw above is $H = \mathrm{Gal}(\ell/k')$. The map $c$ is then restriction from $\mathrm{Gal}(\ell/k)$ to $\mathrm{Gal}(\ell/k')$ followed by the map induced by the inclusion $L^\times \subset \widehat{L}^\times$. The composition $L^\times \to \widehat{L}^\times \xrightarrow{v_1} \mathbf{Z}$ is just the valuation $v_1$ restricted to $L^\times$. Putting all this together, we are reduced to showing that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}^2(\ell/k, L^\times) & \xrightarrow{v_1 \circ \mathrm{Res}} & \mathrm{H}^2(\ell/k', \mathbf{Z}) & \xrightarrow{\mathrm{Infl} \circ \delta^{-1}} & \mathrm{H}^1(\kappa(Z), \mathbf{Q}/\mathbf{Z}) \\
\downarrow{\scriptstyle \mathrm{div}} & & \uparrow & & \\
\mathrm{H}^2(\ell/k, \mathrm{Div}_Z \bar{X}) & \longrightarrow & \mathrm{H}^2(\ell/k, \mathrm{Ind}_{k'/k} \mathbf{Z}) & &
\end{array}.$$

The "top route" on this diagram is our original definition of the residue map, and the "bottom route" is the one used to construct our map $\phi$. That the diagram does indeed commute is easily checked on cocycles. Explicitly, let us start with a cocycle $(\sigma, \tau) \mapsto f_{\sigma\tau}$ lying in $\mathrm{H}^2(\ell/k, L^\times)$. Denote by $v_{Z_1}, \dots, v_{Z_n}$ the valuations on $L^\times$ associated to the divisors $Z_1, \dots, Z_n$. Our chosen cocycle, along the "bottom route", then maps to:

$$\begin{aligned}
(\sigma, \tau) &\mapsto (v_{Z_1}(f_{\sigma\tau}), \dots, v_{Z_n}(f_{\sigma\tau})) \in \mathrm{H}^2(\ell/k, \mathrm{Div}_Z \bar{X}) \\
(\sigma, \tau) &\mapsto (g \mapsto g v_{g^{-1}Z}(f_{\sigma\tau})) \in \mathrm{H}^2(\ell/k, \mathrm{Ind}_{k'/k} \mathbf{Z})
\end{aligned}$$

by Exercise 14.7.8 and Proposition 14.4.7; and to

$$(\sigma, \tau) \mapsto v_1(f_{\sigma\tau}) \in \mathrm{H}^2(\ell/k', \mathbf{Z})$$

by the proof of Shapiro's Lemma (Proposition 14.4.3). This is precisely what we get by following the "top" route. □

**Corollary 15.3.2** *There is an exact sequence*

$$0 \to \mathrm{Br}_1 X \to \mathrm{H}^2(k, \kappa(\bar{X})^\times) \xrightarrow{\mathrm{div}} \mathrm{H}^2(k, \mathrm{Div} \bar{X}).$$

*Proof* As discussed above, $\mathrm{Br}_1 X$ consists of those elements of $\mathrm{Br}\, \kappa(X)$ which are unramified, and lie in the subgroup identified with $\mathrm{H}^2(k, \kappa(\bar{X}))$. The Purity Theorem (Theorem 15.1.11) shows that being unramified is equivalent to having trivial residue at each prime divisor; and Proposition 15.3.1 shows that this is equivalent to being in the kernel of $\mathrm{H}^2(k, \kappa(\bar{X})) \to \mathrm{H}^2(k, \mathrm{Div} \bar{X})$. □

It is often useful to specialise Corollary 15.3.2 to the case of cyclic algebras, as follows.

**Proposition 15.3.3** *Let $X$ be a smooth, geometrically integral variety over a field $k$, $\ell/k$ a finite cyclic extension, $\sigma$ a generator of* $\mathrm{Gal}(\ell/k)$ *and* $f \in \kappa(X)^\times$. *The cyclic algebra* $(\ell/k,\sigma,f)$ *is in the image of the natural map* $\mathrm{Br}(X) \to \mathrm{Br}(\kappa(X))$ *if and only if* $(f) = N_{\ell/k}(D)$*, for some* $D \in \mathrm{Div}(X_\ell)$. *If $k$ is a number field and* $X(k_v) \neq \emptyset$ *for all valuations $v$ of $k$, then* $(\ell/k, f)$ *comes from* $\mathrm{Br}(k)$ *if and only if we can take $D$ to be principal.*

*Proof* □

This description of $\mathrm{Br}_1 X$ as a subgroup of $\mathrm{H}^2(k,\kappa(\bar{X})^\times)$ allows us, with the help of a little more diagram-chasing, to describe a systematic way of producing elements of $\mathrm{Br}_1 X$ in certain situations. Recall that there is a natural map $\mathrm{Br}\,k \to \mathrm{Br}\,\kappa(X)$, the image of which lies in $\mathrm{Br}_1 X$. In view of Remark 13.1.8, if we are interested in calculating the Brauer–Manin obstruction associated to $\mathrm{Br}_1 X$, it suffices to find elements of $\mathrm{Br}_1 X$ which generate the quotient $\mathrm{Br}_1 X/\mathrm{Br}\,k$.

**Proposition 15.3.4** *Let $X$ be a smooth, geometrically irreducible, projective variety over a number field $k$. Then there is an isomorphism*

$$r\colon \mathrm{H}^1(k,\mathrm{Pic}\,\bar{X}) \longrightarrow \mathrm{Br}_1 X/\mathrm{Br}\,k.$$

*Proof* We construct the isomorphism as the composition of two homomorphisms.

**Step 1.** Consider the short exact sequence of Galois modules

$$0 \to \mathrm{Princ}\,\bar{X} \to \mathrm{Div}\,\bar{X} \to \mathrm{Pic}\,\bar{X} \to 0$$

which defines $\mathrm{Pic}\,\bar{X}$. Given that $\mathrm{H}^1(k,\mathrm{Div}\,\bar{X}) = 0$ (from Exercise 14.7.8), the associated long exact sequence in cohomology gives an isomorphism

$$\alpha\colon \mathrm{H}^1(k,\mathrm{Pic}\,\bar{X}) \xrightarrow{\sim} \ker\big(\mathrm{H}^2(k,\mathrm{Princ}\,\bar{X}) \to \mathrm{H}^2(k,\mathrm{Div}\,\bar{X})\big).$$

**Step 2.** By Proposition 4.1.8, the only functions on $\bar{X}$ with trivial divisor are the constant functions. It follows that there is a short exact sequence

$$0 \to \bar{k}^\times \to \kappa(\bar{X})^\times \xrightarrow{\mathrm{div}} \mathrm{Princ}\,\bar{X} \to 0.$$

Part of the associated long exact sequence in cohomology is

$$\mathrm{Br}\,k = \mathrm{H}^2(k,\bar{k}^\times) \to \mathrm{H}^2(k,\kappa(\bar{X})^\times) \xrightarrow{\beta} \mathrm{H}^2(k,\mathrm{Princ}\,\bar{X}) \to \mathrm{H}^3(k,\bar{k}^\times).$$

**Step 3.** We claim that restricting $\beta$ to $\mathrm{Br}_1 X$ gives an isomorphism

$$\beta\colon \mathrm{Br}_1 X/\mathrm{Br}\, k \longrightarrow \ker\left(\mathrm{H}^2(k, \mathrm{Princ}\,\bar{X}) \to \mathrm{H}^2(k, \mathrm{Div}\,\bar{X})\right).$$

Consider the composition of homomorphisms $\kappa(\bar{X})^\times \xrightarrow{\mathrm{div}} \mathrm{Princ}\,\bar{X} \xrightarrow{i} \mathrm{Div}\,\bar{X}$, the second homomorphism being the inclusion map. These give rise to homomorphisms

$$\mathrm{H}^2(k, \kappa(\bar{X})^\times) \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Princ}\,\bar{X}) \xrightarrow{i_*} \mathrm{H}^2(k, \mathrm{Div}\,\bar{X})$$

to which we can apply the kernel-cokernel exact sequence (**??**). We obtain an exact sequence

$$0 \to \ker\beta \to \ker(i_* \circ \beta) \xrightarrow{\beta} \ker i_* \to \mathrm{coker}\,\beta,$$

which translates into

$$\mathrm{Br}\, k \to \mathrm{Br}_1 X \xrightarrow{\beta} \ker\left(\mathrm{H}^2(k, \mathrm{Princ}\,\bar{X}) \to \mathrm{H}^2(k, \mathrm{Div}\,\bar{X})\right) \to \mathrm{H}^3(k, \bar{k}^\times).$$

(We have removed the left-hand 0 because $\ker\beta$ is the *image* of $\mathrm{Br}\, k$ in $\mathrm{H}^2(k, \kappa(\bar{X})^\times)$, rather than $\mathrm{Br}\, k$ itself.) As $k$ is a number field, we have $\mathrm{H}^3(k, \bar{k}^\times) = 0$ by **??**. Therefore $\beta$ does give the claimed isomorphism.

**Step 4.** Define $r := \beta^{-1} \circ \alpha$ to obtain the required isomorphism. □

As we shall describe in more detail in the following section, it is not only the statement of this proposition that is useful, but also the explicit nature of its proof. By following the steps of the proof, we can turn an explicitly given cocycle in $\mathrm{H}^1(k, \mathrm{Pic}\,\bar{X})$ into an explicit central simple algebra over $\kappa(X)$.

## 15.4 Computing the algebraic Brauer group

In this section, we apply the understanding of the algebraic Brauer group gained in the previous section to writing down explicit elements of $\mathrm{Br}_1 X$, for suitable varieties $X$. Other presentations of this material can be found in Kresch and Tschinkel (2008) and Bright and Swinnerton-Dyer (2004).

What exactly is the goal? Let $k$ be a number field and $X$ a smooth, projective, geometrically irreducible variety over $k$. An element of the algebraic Brauer group is a central simple algebra over $\kappa(X)$ split by a finite extension $\ell/k$. Using the explicit correspondence between central simple algebras and 2-cocycles, to "write down" such a central simple algebra it is enough to write down a 2-cocycle representing a class in $\mathrm{H}^2(\ell/k, \kappa(X_\ell)^\times)$. This is a concrete, finite amount of information: writing $G = \mathrm{Gal}(\ell/k)$, a 2-cocycle specifies an element of $\kappa(X_\ell)^\times$ for each pair $(\sigma, \tau)$ of elements of the finite group $G$.

When can we expect to write down "the whole" algebraic Brauer group? Again, suppose that $X$ is a smooth, geometrically irreducible variety over a number field $k$. Then $\mathrm{Br}_1 X$ is infinite, since it contains at least the classes of all the constant algebras; even if $\mathrm{Br}\,k \to \mathrm{Br}_1 X$ is not injective, the kernel is finite and so the image is still very large. On the other hand, constant algebras do not contribute to the Brauer–Manin obstruction, so we are really interested in the quotient $\mathrm{Br}_1 X / \mathrm{Br}\,k$. This does stand some chance of being finite. As we have seen in Proposition 15.3.4, the quotient $\mathrm{Br}_1 X / \mathrm{Br}\,k$ is isomorphic to $\mathrm{H}^1(k, \mathrm{Pic}\,\bar{X})$; the structure of this group depends on the structure of $\mathrm{Pic}\,\bar{X}$, which is a geometric property of the variety $\bar{X}$.

If $X$ is a smooth, projective, geometrically irreducible curve over a number field, then there are two possibilities: either $X$ has genus zero, in which case $\mathrm{Pic}\,\bar{X}$ is isomorphic to $\mathbf{Z}$ with trivial Galois action, and $\mathrm{H}^1(k, \mathrm{Pic}\,\bar{X})$ is trivial; or $X$ has genus at least 1, in which case $\mathrm{Pic}\,\bar{X}$ contains the Abelian variety $\mathrm{Pic}^0\,\bar{X}$. In the latter case, $\mathrm{H}^1(k, \mathrm{Pic}\,\bar{X})$ contains much deep information about the arithmetic of $X$, and in particular contains the Tate–Shafarevich group of the Jacobian of $X$; this is notoriously difficult to understand even in individual examples, so we should not expect to be able to do anything about computing it in general.

On the other hand, surfaces behave very differently. We have seen in this book several examples of surfaces, in particular del Pezzo surfaces and K3 surfaces, which have *finitely generated* Picard group. Galois cohomology with values in a finitely generated abelian group is something we can successfully compute with.

**Proposition 15.4.1** *Let $X$ be a smooth, projective, geometrically irreducible variety over a number field $k$, and suppose that* $\mathrm{Pic}\,\bar{X}$ *is free and finitely generated. Then* $\mathrm{H}^1(k, \mathrm{Pic}\,\bar{X})$*, and therefore* $\mathrm{Br}_1 X / \mathrm{Br}\,k$*, are finite.*

*Proof* Let $Z_1, \dots, Z_n$ be divisors on $\bar{X}$ whose classes freely generate $\mathrm{Pic}\,\bar{X}$. Each divisor $Z_i$ is defined by a finite number of polynomial equations, each of which has finitely many coefficients; there is therefore a finite extension $K/k$ containing all these coefficients, so that every $Z_i$ is defined over $K$. It follows that $\mathrm{Gal}(\bar{k}/K)$ fixes each $Z_i$ and so acts trivially on $\mathrm{Pic}\,\bar{X}$. We can identify $\mathrm{Pic}\,\bar{X}$ with $\mathrm{Pic}\,X_K$. Therefore we have

$$\mathrm{H}^1(K, \mathrm{Pic}\,\bar{X}) = \mathrm{Hom}(\mathrm{Gal}(\bar{k}/K), \mathrm{Pic}\,\bar{X}) = 0$$

by Example 14.7.5. The inflation-restriction sequence now gives

$$0 \to \mathrm{H}^1(K/k, \operatorname{Pic} \bar{X}) \to \mathrm{H}^1(k, \operatorname{Pic} \bar{X}) \to \mathrm{H}^1(K, \operatorname{Pic} \bar{X})$$

and so an isomorphism $\mathrm{H}^1(K/k, \operatorname{Pic} \bar{X}) \cong \mathrm{H}^1(k, \operatorname{Pic} \bar{X})$. This group is finite by Proposition 14.4.11. By Proposition 15.3.4, $\operatorname{Br}_1 X / \operatorname{Br} k$ is finite as well. □

## Computing $\mathrm{H}^1(k, \operatorname{Pic} \bar{X})$

Let $X$ be a variety satisfying the conditions of Proposition 15.4.1. Then not only is $\mathrm{H}^1(k, \operatorname{Pic} \bar{X})$ finite, but we can compute it, in the following sense. Suppose that we are given equations for $X$ and equations for finitely many divisors $Z_1, \dots, Z_n$ on $\bar{X}$, the classes of which freely generate $\operatorname{Pic} \bar{X}$. Suppose further that we already know a finite extension $K/k$ over which all the $Z_i$ are defined, together with $G = \operatorname{Gal}(K/k)$ and its action on the $Z_i$. Then there is an algorithm which computes the structure of $\mathrm{H}^1(k, \operatorname{Pic} \bar{X})$ as a finite group, and also explicit 1-cocycles representing classes which generate the group. (What is an explicit 1-cocycle in this context? It is a collection $\{D_\sigma : \sigma \in G\}$, where each $D_\sigma$ is a class in $\operatorname{Pic} \bar{X}$, represented as a finite integer linear combination of the classes of the $Z_i$.) There is nothing mysterious about this calculation. The groups of cochains $C^i(G, \operatorname{Pic} \bar{X})$ are finitely generated $\mathbf{Z}$-modules, and the coboundary maps between them are linear maps represented by integer matrices, which can be written down. Computing the cohomology groups is therefore a matter of integer linear algebra, for which algorithms are known. (In practice there may well be more efficient ways of computing cohomology groups, but at least we can see that it is a finite calculation.)

As a slight extension of this, we do not necessarily require that the $Z_i$ generate $\operatorname{Pic} \bar{X}$ *freely*; it is enough to have a finite set of divisors which generate $\operatorname{Pic} \bar{X}$. We will need to know the relations between their classes, that is, the kernel of the map $\oplus_i Z_i \to \operatorname{Pic} \bar{X}$. Under the hypothesis that $\operatorname{Pic} \bar{X}$ is free, numerical equivalence of divisors is the same as linear equivalence, and so this kernel can be deduced from the intersection numbers of the $Z_i$.

## Shrinking the field extension

Keeping the notation of the previous section, suppose that we have been given the variety $X$, the divisors $Z_i$ which generate $\operatorname{Pic} \bar{X}$, the field $K$ over which the $Z_i$ are all defined, and the Galois group $\operatorname{Gal}(K/k)$; we have now computed $\mathrm{H}^1(k, \operatorname{Pic} \bar{X})$. The proof of Proposition 15.3.4 shows how to turn a class in $\mathrm{H}^1(k, \operatorname{Pic} \bar{X})$ into a central simple algebra over $\kappa(X)$ representing the corresponding class in $\operatorname{Br}_1 X$. At the moment any class in $\mathrm{H}^1(k, \operatorname{Pic} \bar{X})$ is represented by a cocycle lying in $\mathrm{H}^1(K/k, \operatorname{Pic}\bar{X})$. However, the extension $K/k$ may well be quite large, making calculations in cohomology (and in particular the procedure in the proof of Proposition 15.3.4) prohibitively lengthy. By definition we have

$$\mathrm{H}^1(k, \operatorname{Pic}\bar{X}) = \varinjlim \mathrm{H}^1(\ell/k, (\operatorname{Pic}\bar{X})^{\mathrm{Gal}(\bar{k}/\ell)})$$

where $\ell$ runs over all finite extensions of $k$, the maps being the inflation maps. Therefore any class in $\mathrm{H}^1(k, \operatorname{Pic}\bar{X})$ arises by inflation from a class in some $\mathrm{H}^1(\ell/k, (\operatorname{Pic}\bar{X})^{\mathrm{Gal}(\bar{k}/\ell)})$. We have already shown that this is true with $\ell = K$, but for what follows it will be useful to be able to take $[\ell : k]$ as small as possible.

This can be accomplished as follows. We already have $\mathrm{H}^1(k, \operatorname{Pic}\bar{X}) \cong \mathrm{H}^1(G, \operatorname{Pic}\bar{X})$ where $G = \mathrm{Gal}(K/k)$. Fix a cocycle $\phi$ representing a class in $\mathrm{H}^1(G, \operatorname{Pic}\bar{X})$. Let $G'$ be a normal subgroup of $G$. By the same procedure as before, we can compute $\mathrm{H}^1(G/G', (\operatorname{Pic}\bar{X})^{G'})$ and the inflation map

$$\mathrm{H}^1(G/G', (\operatorname{Pic}\bar{X})^{G'}) \xrightarrow{\mathrm{Infl}} H^1(G, \operatorname{Pic}\bar{X})$$

Trying all subgroups $G'$, we can find the largest $G'$ such that $\phi$ lies in the image of the inflation map, and therefore (setting $\ell = K^{G'}$) the smallest extension $\ell/k$ such that the class of $\phi$ arises by inflation from $\mathrm{H}^1(\ell/k, (\operatorname{Pic}\bar{X})^{\mathrm{Gal}(\bar{k}/\ell)}$.

There remains one problem. In what follows, we will show how to convert an explicit cocycle in $\mathrm{H}^1(\ell/k, \operatorname{Pic}X_\ell)$ into a central simple algebra in $\mathrm{Br}_1 X$. However, instead of a cocycle with values in $\operatorname{Pic}X_\ell$, we have in our hands at this stage a cocycle with values in $(\operatorname{Pic}\bar{X})^{\mathrm{Gal}(\bar{k}/\ell)}$, or in other words a cocycle with values in $(\operatorname{Pic}X_K)^{G'}$. In detail, an element of $(\operatorname{Pic}X_K)^{G'}$ is represented a divisor (in fact, a linear combination of the divisors $Z_i$ which we started with) which is linearly equivalent to each of its conjugates under the action of $G'$. Assuming that $X$ has points everywhere locally, the method described in Section **??** shows how to find an equivalent divisor defined over $\ell$, that is, to make effective the isomorphism $(\operatorname{Pic}X_K)^{G'} \cong \operatorname{Pic}X_\ell$.

After following this procedure, we now have: a description of the group $\mathrm{H}^1(k, \operatorname{Pic}\bar{X})$; and, for each class $c$ in that group, a finite extension $\ell/k$ and an explicit cocycle in $\mathrm{H}^1(\ell/k, \operatorname{Pic}X_\ell)$ representing the class $c$.

### From $\mathrm{H}^1(k, \operatorname{Pic}\bar{X})$ to $\mathrm{Br}_1 X$

Suppose now that we are given a finite Galois extension $\ell/k$, with known Galois group $H$, and an explicit 1-cocycle $\phi\colon H \to \operatorname{Pic}X_\ell$ representing a class in $\mathrm{H}^1(H, \operatorname{Pic}X_\ell)$. We assume that, for each $\sigma \in H$, $\phi(\sigma)$ is specified as a divisor on $X_\ell$, the class of which is $\phi(\sigma)$. The recipe of Proposition 15.3.4 consists of the following steps.

**Step 1.** Compute the image of $\phi$ under the connecting homomorphism $\alpha\colon \mathrm{H}^1(H, \mathrm{Pic}\, X_\ell) \to \mathrm{H}^2(H, \mathrm{Princ}\, X_\ell)$. This is a straightforward application of the definition. Firstly, we lift $\phi$ to a map $H \to \mathrm{Div}\, X_\ell$; this amounts to doing nothing, since each $\phi(\sigma)$ was already given as a divisor. Secondly, we define a 2-chain $\psi = \alpha(\phi)$ by

$$\psi(\sigma, \tau) = \phi(\sigma) + \sigma\phi(\tau) - \phi(\sigma\tau).$$

Then $\psi$ is actually a 2-cocycle taking values in $\mathrm{Princ}\, X_\ell$.

**Step 2.** Try to lift $\psi$ under the homomorphism $\beta\colon \mathrm{H}^2(H, \kappa(X_\ell)^\times) \to \mathrm{H}^2(H, \mathrm{Princ}\, X_\ell)$. There are potential problems here, since we must make use of the triviality of $\mathrm{H}^3(k, \bar{k}^\times)$. The process is as follows. Firstly, for each pair $(\sigma, \tau)$ of elements of $H$, find a rational function $f_{\sigma,\tau} \in \kappa(X_\ell)$ such that $(f_{\sigma,\tau}) = \psi(\sigma, \tau)$. (This is possible since $\psi(\sigma, \tau)$ is a principal divisor, by construction.[1]) Now the map $\eta\colon (\sigma, \tau) \mapsto f_{\sigma,\tau}$ is a 2-cochain with values in $\kappa(X_\ell)^\times$, but it has no reason to be a cocycle. Indeed, the $f_{\sigma,\tau}$ are each unique only up to a constant multiple, and we have no canonical way of choosing them. To see how far $\eta$ is from being a cocycle, we look at the coboundary $\xi = \partial\eta$, which is a 3-cocycle taking values in $\ell^\times$. The fact that $\mathrm{H}^3(k, \bar{k}^\times)$ is trivial unfortunately does not imply that $\xi$ is a coboundary, because $\mathrm{H}^3(\ell/k, \ell^\times)$ might not be trivial. What is true is that there is some finite extension $\ell'/\ell$ such that the class of $\xi$ maps to zero in $\mathrm{H}^3(\ell'/k, \ell'^\times)$ under the inflation map. Indeed, there is an effective algorithm to find such an $\ell'$, and to compute a 2-cochain $\zeta \in C^2(\mathrm{Gal}(\ell'/k), \ell'^\times)$ such that $\partial\zeta = \mathrm{Infl}(\xi)$: see Kresch and Tschinkel (2008, Proposition 6.3). Replacing $\ell$ by $\ell'$ and $\eta$ by $\eta/\zeta$, we reach the stage where $\eta$ is indeed a 2-cocycle with values in $\kappa(X_\ell)^\times$. By the proof of Proposition 15.3.4, $\eta$ represents the class of our desired algebra in $\mathrm{Br}_1 X$.

*Remark* 15.4.2 If $X$ has a rational point over $k$, then enlarging $\ell$ to $\ell'$ will never be necessary. The reason is that a rational point $P \in X(k)$ gives a way of choosing the $f_{\sigma,\tau}$ correctly in the first place: assuming that $P$ does not lie on the support of any of the divisors appearing in $\psi$, scale such that $f_{\sigma,\tau}(P)$ is 1.

[1] The curious reader may wonder how to find a rational function corresponding to a principal divisor. Essentially the algorithm works as follows. We work in projective space. Given a divisor known to be principal, write it as $D^+ - D^-$ with $D^+, D^-$ effective. For some sufficiently large positive integer $d$, there is a form $g$ of degree $d$ which vanishes on $D^-$; but $g$ will also vanish elsewhere, say on another divisor $E$. Now, if $d$ was chosen sufficiently large, there will be another form $f$, also of degree $d$, vanishing precisely on $D^+$ and $E$; the function $f/g$ then satisfies $(f/g) = D^+ - D^-$. For more details, see Kresch and Tschinkel (2008, Proposition 7.3).

Then $\eta$ is automatically a cocycle. (Even if $P$ does lie on one of the divisors, the implication still holds: see Skorobogatov (2001, Theorem 2.3.4(b)) and the proof at the bottom of p. 27 there.) Turning this around, this means that if $\xi$ is not already a coboundary in $\mathrm{H}^3(\ell/k, \ell^\times)$, then $X$ has no rational points over $k$. If the goal of computing the Brauer group was to try to find an obstruction to the existence of rational points on $X$, then there is no need to continue.

## The case of a cyclic extension

An important special case of the calculation described in the preceding section is when the extension $\ell/k$ is cyclic. Recall from Section 14.5 that, for a finite cyclic group $G$ acting on a module $M$, there are isomorphisms

$$\mathrm{H}^1(G, M) \cong {}_{\mathrm{N}}M/\Delta M \quad \text{and} \quad \mathrm{H}^2(G, M) \cong M^G/\mathrm{N}M.$$

It is important to keep in mind that these isomorphisms are not canonical, but depend on the choice of a generator of $G$.

We keep the notation of the preceding section: $X$ is a smooth, geometrically irreducible variety over a number field $k$, and $\ell/k$ is a finite extension. Now suppose that $\mathrm{Gal}(\ell/k)$ is cyclic, and fix a generator $\sigma$ of $\mathrm{Gal}(\ell/k)$. We have the isomorphism

$$\mathrm{H}^1(\ell/k, \mathrm{Pic}\, X_\ell) \cong {}_{\mathrm{N}}(\mathrm{Pic}\, X_\ell)/\Delta \mathrm{Pic}\, X_\ell.$$

By Proposition 15.3.4, every element of ${}_{\mathrm{N}}(\mathrm{Pic}\, X_\ell)/\Delta \mathrm{Pic}\, X_\ell$ gives rise to a class in $\mathrm{Br}_1 X/\mathrm{Br}\, k$; we would like to describe this class without going through the $\mathrm{H}^1$s and $\mathrm{H}^2$s of the procedure described previously.

**Proposition 15.4.3** *Let $D \in \mathrm{Div}\, X_\ell$ be a divisor representing a class in ${}_{\mathrm{N}}(\mathrm{Pic}\, X_\ell)/\Delta \mathrm{Pic}\, X_\ell$. The divisor $\mathrm{N}D \in \mathrm{Div}\, X$ is principal; let $f \in \kappa(X)^\times$ be such that $(f) = \mathrm{N}D$. Then the image of the class of $D$ under the sequence of maps*

$${}_{\mathrm{N}}(\mathrm{Pic}\, X_\ell)/\Delta \mathrm{Pic}\, X_\ell \cong \mathrm{H}^1(\ell/k, \mathrm{Pic}\, X_\ell) \xrightarrow{\mathrm{Infl}} \mathrm{H}^1(k, \mathrm{Pic}\, \bar{X}) \xrightarrow{r} \mathrm{Br}_1 X/\mathrm{Br}\, k$$

*is the class of the cyclic algebra $(\kappa(X_\ell)/\kappa(X), \sigma, f)$.*

*Proof* By the various functorial properties described in Proposition 14.5.3, the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}^1(k, \operatorname{Pic} \bar{X}) & \xrightarrow{\alpha} & \mathrm{H}^2(k, \operatorname{Princ} \bar{X}) & \xleftarrow{\beta} & \mathrm{H}^2(k, \kappa(\bar{X})^\times) \\
\mathrm{Infl}\big\uparrow & & \mathrm{Infl}\big\uparrow & & \mathrm{Infl}\big\uparrow \\
\mathrm{H}^1(\ell/k, \operatorname{Pic} X_\ell) & \longrightarrow & \mathrm{H}^2(\ell/k, \operatorname{Princ} X_\ell) & \longleftarrow & \mathrm{H}^2(\ell/k, \kappa(X_\ell)^\times) \cdot \\
\cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow \\
{}_{\mathrm{N}}\operatorname{Pic} X_\ell / \Delta \operatorname{Pic} X_\ell & \xrightarrow{\mathrm{N}_*} & \operatorname{Princ} X / \mathrm{N}(\operatorname{Princ} X_\ell) & \xleftarrow{\mathrm{div}} & \kappa(X)^\times / \mathrm{N}\kappa(X_\ell)^\times
\end{array}$$

The map $\beta^{-1} \circ \alpha$ of the top row was the definition of the isomorphism $r$ of Proposition 15.3.4, and so the sequence of maps described in the statement of the proposition takes us up the left-hand column of the diagram and across the top row. On the other hand, going along the bottom row of the diagram takes our divisor $D$ to the class of $f$ in $\kappa(X)^\times / N\kappa(X_\ell)^\times$, and then **??** shows that going up the right-hand column maps the class of $f$ to the class of the cyclic algebra $(\kappa(X_\ell)/\kappa(X), \sigma, f)$, as claimed. □

To summarise, the procedures described in this section allow us to do the following. Suppose that we are given a smooth, projective, geometrically irreducible variety $X$ over a number field $k$, together with a finite Galois extension $K/k$ and divisors $Z_1, \dots, Z_n$, defined over $K$, the classes of which generate $\operatorname{Pic} \bar{X}$, which we suppose to be free. Suppose that we know $\operatorname{Gal}(K/k)$ and its action on the $Z_i$. Then we can compute the abstract structure of $\operatorname{Br}_1 X / \operatorname{Br} k$; and, for each class in $\operatorname{Br}_1 X / \operatorname{Br} k$, we can find a minimal finite extension $\ell/k$ contained in $K$ and an explicit 2-cocycle in $\mathrm{H}^2(\ell/k, \kappa(X_\ell)^\times)$ representing that class. If $\ell/k$ is cyclic, we can instead realise the cohomology class as the class of an explicit cyclic algebra.

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