Computations on diagonal quartic surfaces

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Summary

We are concerned with the solubility in integers of the equation

$$a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0$$

where a_0, a_1, a_2 and a_3 are fixed integral coefficients. As the equation is homogeneous, this is equivalent to studying rational solutions. This equation defines a surface in three-dimensional projective space $\mathbb{P}^3$, which we denote by V. Geometrically, V is a K3 surface.

One condition which is clearly necessary for V to have rational points is that there be points on V in each completion $\mathbb{Q}_v$ of $\mathbb{Q}$. Varieties for which this condition is sufficient are said to satisfy the *Hasse principle*. In general, the Hasse principle does not hold, but many counterexamples are explained by the so-called *Brauer–Manin obstruction*. This was first put forward by Manin and is defined in terms of the *Brauer group* of the variety.

The aim of this dissertation is to classify all diagonal quartic surfaces, defined by the above equation, into finitely many cases. The cases are distinguished by the action of the absolute Galois group of $\mathbb{Q}$ on the Picard group of V, and we show that there are 546 such cases. They are described in terms of algebraic constraints on the coefficients a_i. In each case, we compute that part of the Brauer group of V which splits over the algebraic closure of $\mathbb{Q}$; this is done using the Galois cohomology of the Picard group of V. We also investigate the elliptic fibrations which may exist on V: to such a fibration is associated a *vertical Brauer group*, and we show how these groups may also be computed. Finally, an indication is given of how these results may be used to look further into the Brauer–Manin obstruction on V.