Efficient calculation of the Brauer–Manin obstruction

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Why are we interested?

- Calculating the Manin obstruction can show that certain Diophantine equations have no solutions.
- Being able to calculate the obstruction efficiently allows us to search through many cases and come up with examples of certain types of behaviour.
- Investigating *how* to calculate the obstruction efficiently gives theoretical insight, which again guides us when looking for examples of certain types of behaviour.

Let X be a smooth complete variety over the rational numbers.

An **Azumaya algebra** on X is a family of central simple algebras over $\mathbb{Q}$ parametrised by the points of X.

One way of looking at an Azumaya algebra is as a special type of central simple algebra over the function field $k(X)$. Such algebras, modulo the usual equivalence relation, are elements of the Brauer group of $k(X)$.

An element of $\mathrm{Br}\, k(X)$ is an Azumaya algebra on X if it is "unramified" in a certain sense. We must be able to specialise the algebra at each point of X.

In this talk we will be concerned only with Azumaya algebras split by a quadratic extension K of $\mathbb{Q}$. Such an algebra is a quaternion algebra over $k(X)$, given by

$$\mathcal{A} = (K/\mathbb{Q}, f)$$

where $f \in k(X)^\times$ has divisor of the form

$$(f) = D + D^\sigma,$$

D being a divisor on X_K, and σ the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$.

The **constant algebras** are those which come from $\mathrm{Br}\,\mathbb{Q}$. Our algebra is equivalent to a constant one if D is principal; replacing D by a linearly equivalent divisor changes $\mathcal{A}$ only by a constant algebra.

An Azumaya algebra $\mathcal{A}$ can be specialised at a point x of $X(\mathbb{Q})$ to obtain a central simple algebra $\mathcal{A}(x) \in \mathsf{Br}\,\mathbb{Q}$. Similarly, if $x_v \in X(\mathbb{Q}_v)$ is a point over some completion of $\mathbb{Q}$, we get $\mathcal{A}(x_v) \in \mathsf{Br}\,\mathbb{Q}_v$.

Class field theory gives an exact sequence

$$0 \to \mathsf{Br}\,\mathbb{Q} \to \bigoplus_v \mathsf{Br}\,\mathbb{Q}_v \xrightarrow{\sum \mathsf{inv}_v} \mathbb{Q}/\mathbb{Z}$$

where the inv_v are the local invariant maps at each place v. At an adèlic point $(x_v) \in \prod_v X(\mathbb{Q}_v)$, we may compute the sum

$$\sum_v \mathsf{inv}_v\, \mathcal{A}(x_v).$$

Manin observed that the adèlic point can only come from a rational point if this sum is zero.

To evaluate the Brauer–Manin obstruction associated to an Azumaya algebra $\mathcal{A}$, we must calculate the sum

$$\sum_v \mathrm{inv}_v \, \mathcal{A}(x_v).$$

at each adèlic point (x_v) on X. This is made possible by two facts:

- For all but finitely many v, the local invariant $\mathrm{inv}_v \, \mathcal{A}(x_v)$ is zero for all x_v in $X(\mathbb{Q}_v)$.

- For all v, the function

  $$x_v \mapsto \mathrm{inv}_v \, \mathcal{A}(x_v)$$

  is continuous, hence locally constant, on $X(\mathbb{Q}_v)$.

This leads to an obvious algorithm for computing the obstruction. At each finite prime where the invariant is not known to be everywhere zero, we list the points on $X(\mathbb{Z}/p\mathbb{Z})$. Sometimes this will be enough to compute the invariant; if not, we look at all lifts of these points to $X(\mathbb{Z}/p^2\mathbb{Z})$, then $X(\mathbb{Z}/p^3\mathbb{Z})$, and so on.

For our quaternion algebra $(K/\mathbb{Q}, f)$, the local invariant at any point is either 0 or $\frac{1}{2}$. If at some v it takes both values 0 and $\frac{1}{2}$ on $X(\mathbb{Q}_v)$, there can be no Brauer–Manin obstruction.

To evaluate the invariant where f has a zero or pole, we switch to another equivalent algebra by replacing D with a linearly equivalent divisor.

At which primes is the invariant everywhere zero?

Theorem 1. *Let* p *be a prime which does not divide the function* f*, where* X *has smooth reduction, and where the extension* $K/\mathbb{Q}$ *is unramified. Then* $\mathrm{inv}_p \mathcal{A}(x_p) = 0$ *for all points* x_p *in* $X(\mathbb{Q}_p)$*.*

By looking at the proof of this theorem, we can extend it to give very useful information at other primes too. We will use this to greatly reduce the amount of work needed to calculate the invariants at those other primes.

This is how the theorem may be proved.

- Extend x_p to a $\mathbb{Z}_p$-point of X (since X is proper). Let $\tilde{x}$ be the reduction of this point modulo p.

- Choose the divisor D so that the reduction of f to $X(\mathbb{F}_p)$ is non-zero near $\tilde{x}$.

- As $K/\mathbb{Q}$ is unramified at p, the invariant map at p factors through the valuation map:

$$H^2(K/\mathbb{Q}, K^\times) \xrightarrow{v_p} H^2(K/\mathbb{Q}, \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

  and so the invariant is zero near $\tilde{x}$.

What can we say at other odd primes p?

- If p splits in $K/\mathbb{Q}$, the Azumaya algebra becomes trivial over $\mathbb{Q}_p$ and the invariant is everywhere zero.

- If $K/\mathbb{Q}$ is unramified at p, the theorem still applies on that part of $X(\mathbb{Q}_p)$ which has smooth reduction.

- If $K/\mathbb{Q}$ is ramified at p, we can still choose f to be a p-adic unit at any point having smooth reduction. Moreover, Galois theory shows that $(\tilde{f}) = 2\tilde{D}$.

Example. Let X be the surface in $\mathbb{P}^3$ defined by the integer equation

$$a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0.$$

If $a_0a_1a_2a_3$ is not a rational square, then there is a non-trivial Azumaya algebra on X given by $(K/\mathbb{Q}, f)$, where

$$K = \mathbb{Q}(\sqrt{a_0a_1a_2a_3}) \text{ and } f = D + D^\sigma$$

where D is a curve of genus 1 minus a plane section.

The odd primes where X has bad reduction are those dividing $a_0a_1a_2a_3$; those where the algebra is ramified are those doing so to an odd power.

If p is a prime dividing precisely one of the a_i, then the reduction of X at p is a cone over a smooth curve of genus 3. Moreover, the function f defining the Azumaya algebra can be chosen so that its reduction is a pull-back from this curve.

- If $K/\mathbb{Q}$ is unramified at p, the invariant is guaranteed to be zero except possibly at points lying over the vertex of the cone. But that point does not lift to $X(\mathbb{Q}_p)$, so the invariant is zero everywhere.

- If $K/\mathbb{Q}$ is ramified at p, we only need to evaluate the algebra at each point of a curve, rather than at each point of a surface.

If p is a prime dividing precisely two of the a_i, then the reduction of X at p consists of four planes all meeting in one common line. In this case, if $K/\mathbb{Q}$ is ramified at p, then

$$(\tilde{f}) = 2\tilde{D} \text{ and so } \tilde{f} = ag^2$$

for some $a \in \mathbb{F}_p^\times$ and some function g. The invariant is therefore constant on $X(\mathbb{Q}_p)$ and it is only necessary to evaluate it at one point.

Note that, for a Brauer–Manin obstruction to exist, at each prime p where $K/\mathbb{Q}$ is ramified, the function f must be either always square or always non-square. Naïvely, it would seem that this is far easier to achieve when p divides two a_i (the invariant is constant) than when p divides only one a_i (when we evaluate the function on a curve). This is borne out by experiment.

Example. Let X be the surface

$$9X_0^4 + 10X_1^4 = 12X_2^4 + 13X_3^4.$$

An Azumaya algebra on X is given by $(K/\mathbb{Q}, f)$ where

$$K = \mathbb{Q}(\sqrt{390}) \text{ and } f = \frac{15X_0^2 + 12X_2^2 + 13X_3^2}{X_0^2}.$$

At $p = 3$, X reduces to four planes and f to the form $(X_3/X_0)^2$, so the invariant is everywhere zero.

At $p = 5$ and $p = 13$, X reduces to a cone over a diagonal quartic curve. In each case, the reduction of f takes both square and non-square values on the reduction of X, so there is no Brauer–Manin obstruction.

Example (Birch–Swinnerton-Dyer, 1975). Let X be the surface in $\mathbb{P}^4$ defined by the two equations

$$\begin{aligned} uv &= x^2 - 5y^2, \\ (u+v)(u+2v) &= x^2 - 5z^2. \end{aligned}$$

There is only one class of non-trivial Azumaya algebras on X, and a representative is $(K/\mathbb{Q}, f)$ where

$$K = \mathbb{Q}(\sqrt{5}) \text{ and } f = \frac{u}{u+v}.$$

At all primes apart from 5, an elementary argument shows that the invariant is everywhere zero.

At $p = 5$, the reduction of X is defined by

$$uv = (u + v)(u + 2v) = x^2$$

which is a scheme of dimension 0 with only two points defined over $\mathbb{F}_5$. The value of f is a non-square at both of these points, so the local invariant is everywhere $\frac{1}{2}$ on $X(\mathbb{Q}_5)$. Hence there is a Brauer–Manin obstruction, and the surface has no rational points.