Computing Brauer–Manin obstructions on diagonal quartic surfaces

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Outline

1. Introduction
   - The Hasse principle
   - The Brauer group
   - The Brauer–Manin obstruction
2. Computing the Brauer–Manin obstruction
   - Computing the algebraic Brauer group
   - Finding the Azumaya algebras
   - Magma demo
3. Theoretical results on the evaluation map
   - Smooth models
   - Unramified places
   - Tamely ramified places

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The Hasse principle

- Let X be a variety over a number field k. Write $\mathbb{A}_k$ for the ring of adèles of k. The set of adelic points of X is $X(\mathbb{A}_k)$; the set of rational points $X(k)$ is contained in it, under the diagonal embedding. If X is a complete variety, then

$$X(\mathbb{A}_k) = \prod_v X(k_v)$$

where the product is over all places v of k.

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$$X(\mathbb{A}_k) = \prod_v X(k_v)$$

  where the product is over all places v of k.
- Some classes of varieties satisfy the *Hasse principle*: that is,

$$X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset.$$

  In this case, it is straightforward to decide whether X has rational points, since the condition on the left is a finite computation.

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  has points in every completion of $\mathbb{Q}$, but no rational points.
- Manin showed that one can use the Brauer group of X to define a subset of $X(\mathbb{A}_k)$ which must contain $X(k)$. If this set is empty, we say that there is a Brauer–Manin obstruction to the Hasse principle for X. This accounted for all counterexamples to the Hasse principle known then.

The Brauer group of the function field

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- We might hope to be able to evaluate an element of $\mathrm{Br}\,k(X)$ at a point of X, to obtain an element of $\mathrm{Br}\,k$.
- Just as a rational function cannot be evaluated at every point of a variety, so a typical element of $\mathrm{Br}\,k(X)$ cannot be evaluated everywhere – it is ramified along some divisor.

The Brauer group of a variety

- Let X be a smooth, geometrically irreducible variety over k. The Brauer group of X, written $\mathrm{Br}\,X$, can be informally defined as the subgroup of $\mathrm{Br}\,k(X)$ of those elements which can be evaluated everywhere. These algebras are called Azumaya algebras.

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- We will be interested only in algebraic elements of $\operatorname{Br} X$, that is, those which are split by an extension of k. These can be described in Galois cohomology as

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- Equivalently, a class α in $H^2(k, k(\bar{X})^\times)$ lies in $\operatorname{Br}_1 X$ if and only if, for all points $P \in X$, we can represent α by a cocycle taking values in $\mathcal{O}_{X,P}^\times$.

The Brauer group of a variety

Example

Let l/k be a quadratic extension, and suppose that f is a rational function on X whose divisor is a norm from l, say $(f) = N_{l/k}D$. Then the quaternion algebra $\mathcal{A} = (l/k, f)$ is an Azumaya algebra on X.

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- To see this, let P be any point of X. If f is invertible at P, then $\mathcal{A}$ can be evaluated at P to get $\mathcal{A}(P) = (l/k, f(P))$.
- Otherwise, there is some divisor $D' \sim D$ which avoids P; let $(g) = D' - D$. Then the algebra $(l/k, fN_{l/k}g)$ is isomorphic to $\mathcal{A}$ and can be evaluated at P.

The Brauer–Manin obstruction

- Let v be a place of k. Recall from class field theory that there is a canonical injection $\mathrm{inv}_v : \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$, such that the sequence

$$0 \to \operatorname{Br} k \to \bigoplus_v \operatorname{Br} k_v \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z}$$

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- If $\mathcal{A}$ is an Azumaya algebra on X and $P_v \in X(k_v)$, then $\mathcal{A}$ can be evaluated at P_v to get an element of $\mathrm{Br}\, k_v$. So $\mathcal{A}$ gives maps

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- Combining these two facts, we get...

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$$\begin{array}{ccc} X(k) & \longrightarrow & X(\mathbb{A}_k) \\ \mathcal{A}\big\downarrow & & \mathcal{A}\big\downarrow \\ \operatorname{Br} k & & \bigoplus_v \operatorname{Br} k_v \end{array}$$

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- Given a subset B of $\operatorname{Br} X$, define

$$X(\mathbb{A}_k)^B := \Big\{ (P_v) \in X(\mathbb{A}_k) \;\Big|\; \sum_v \operatorname{inv}_v \mathcal{A}(P_v) = 0 \text{ for all } \mathcal{A} \in B \Big\}.$$

We have shown that $X(k) \subset X(\mathbb{A}_k)^{\operatorname{Br} X}$.

Comments

- If $X(\mathbb{A}_k)^B$ is empty, we say there is a Brauer–Manin obstruction to the Hasse principle coming from B. If $X(\mathbb{A}_k)^B$ is not the whole of $X(\mathbb{A}_k)$, we say there is a Brauer–Manin obstruction to weak approximation.

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- We have constant Azumaya algebras $\operatorname{Br} k \subset \operatorname{Br} X$, but the condition they impose is vacuous. So the Brauer–Manin obstruction is determined by $\operatorname{Br} X / \operatorname{Br} k$.
- We will show how to compute generators for the algebraic part, $\operatorname{Br}_1 X / \operatorname{Br} k$, and the associated obstruction.

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Computing the algebraic Brauer group

- Recall that the algebraic part of the Brauer group, $\operatorname{Br}_1 X$, can be described as a Galois cohomology group

$$\operatorname{Br}_1 X = \ker\left(H^2(k, k(\bar{X})^\times) \to H^2(k, \operatorname{Div}\bar{X})\right).$$

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- However, we only need to know generators for $\operatorname{Br}_1 X / \operatorname{Br} k$. Write the homomorphism above as a composition

$$H^2(k, k(\bar{X})^\times) \xrightarrow{f} H^2(k, \operatorname{Princ}\bar{X}) \xrightarrow{g} H^2(k, \operatorname{Div}\bar{X}).$$

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- The kernel-cokernel exact sequence for this composition of maps is

$$0 \to \ker f \to \operatorname{Br}_1 X \to \ker g \to \operatorname{coker} f$$

  and we can identify these groups.

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- Using the exact sequence

$$0 \to \bar{k}^\times \to k(\bar{X})^\times \to \mathrm{Princ}\,\bar{X} \to 0$$

shows that $\ker f = \mathrm{im}(\mathrm{Br}\, k)$, and that $\mathrm{coker}\, f = H^3(k, \bar{k}^\times) = 0$.

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shows that $\ker g$ is the image of the boundary map $\partial : H^1(k, \operatorname{Pic} \bar{X}) \to H^2(k, \operatorname{Princ} \bar{X})$. Since $\operatorname{Div} \bar{X}$ is an induced module, this map is injective.

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- So there is an isomorphism $\operatorname{Br}_1 X / \operatorname{Br} k \cong H^1(k, \operatorname{Pic} \bar{X})$.

Computing the algebraic Brauer group

- We have an isomorphism $\operatorname{Br}_1 X / \operatorname{Br} k \cong H^1(k, \operatorname{Pic} \bar{X})$. If $\operatorname{Pic} \bar{X}$ is finitely generated, then we can hope to understand this group. If $\operatorname{Pic} \bar{X}$ is also free, then $\operatorname{Br}_1 X / \operatorname{Br} k$ is finite.

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- If we know explicitly a finite, Galois-stable set of generators for $\operatorname{Pic} \bar{X}$, and the Galois action on them, then computing $H^1(k, \operatorname{Pic} \bar{X})$ is straightforward.

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- On a diagonal quartic surface, there are 48 straight lines. We can write down their equations, and they generate $\operatorname{Pic} \bar{X}$.
- The Galois group of the field of definition of the 48 lines is always a subgroup of the "generic" Galois group, which is an extension of C_2 by $C_2 \times C_4 \times C_4$. Going through all the possible Galois actions finds all possibilities for $\operatorname{Br}_1 X / \operatorname{Br} k$. It is always killed by 8, and has 2-rank at most 7.

Finding the Azumaya algebras

- Getting our hands on explicit generators for $H^1(k, \operatorname{Pic} \bar{X})$ is only the first step to computing the algebraic Brauer–Manin obstruction. We now need to turn them into explicit generators for $\operatorname{Br}_1 X / \operatorname{Br} k$.

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- The isomorphism $H^1(k, \operatorname{Pic} \bar{X}) \cong \operatorname{Br}_1 X / \operatorname{Br} k$ arose as a composition of various maps:

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- The first of these, ∂, is a boundary map in cohomology and is straightforward to compute: lift from $\operatorname{Pic} \bar{X}$ to $\operatorname{Div} \bar{X}$ and take the coboundary. Note that there is a choice of lifts here, giving different but cohomologous images.

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- Computing g^{-1} involves lifting from $\operatorname{Princ} \bar{X}$ to $k(\bar{X})^\times$, a potentially slow operation. Moreover, lifting just anyhow will not give us a cocycle – to do that, we need to make effective the fact that $H^3(k, \bar{k}^\times) = 0$.

Using a small splitting field

- Some of these problems become easier if the elements of $H^1(k, \operatorname{Pic} \bar{X})$ we're looking at are split by a small extension l/k.

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\sim \uparrow & & \sim \uparrow & & \sim \uparrow \\
\dfrac{{}_N \operatorname{Pic} X_l}{\langle \sigma - 1 \rangle} & \xrightarrow{N} & \dfrac{\operatorname{Princ} X}{N \operatorname{Princ} X_l} & \longleftarrow & \dfrac{k(X)^\times}{N k(X_l)^\times}
\end{array}
$$

- If l/k is cyclic, things get even more straightforward.

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H^1(l/k, \operatorname{Pic} X_l) & \longrightarrow & H^2(l/k, \operatorname{Princ} X_l) & \longleftarrow & H^2(l/k, k(X_l)^\times) \\
\sim \uparrow & & \sim \uparrow & & \sim \uparrow \\
\frac{{}_N \operatorname{Pic} X_l}{\langle \sigma - 1 \rangle} & \xrightarrow{N} & \frac{\operatorname{Princ} X}{N \operatorname{Princ} X_l} & \longleftarrow & \frac{k(X)^\times}{N k(X_l)^\times}
\end{array}
$$

- If l/k is cyclic, things get even more straightforward.
- But we have introduced a new problem: we probably don't know a set of divisors defined over l which generate $\operatorname{Pic} X_l$.

Magma demo

Outline

Theoretical results on the evaluation map

Let $\mathcal{A}$ be an Azumaya algebra on X, and fix a finite place v. We will apply some geometry to understand the evaluation map

$$X(k_v) \to \mathbb{Q}/\mathbb{Z} \qquad P \mapsto \operatorname{inv}_v \mathcal{A}(P).$$

- We saw in the demonstration that, at primes of good reduction, the invariant was everywhere zero. For each $P \in X(k_v)$, we could always find one of our representative algebras $(-1, f)$ such that $f(P)$ was a unit in k_v.

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- Of course, we could spoil this: we could change our algebra by a constant algebra ramified at v. The invariant would still be constant, but not necessarily zero.

Smooth models

It is much easier to investigate the behaviour of $\mathcal{A}(P)$ when P reduces to a smooth point. What does this mean for diagonal quartic surfaces?

- Consider the diagonal quartic surface

$$X \quad : \quad a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0$$

  where $a_i \in \mathbb{Q}$. We may clearly assume that the a_i are coprime integers, and that none of them is divisible by a fourth power. Reducing the equation modulo p gives a surface over $\mathbb{F}_p$ which may be singular.

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- But this is only one model of X; we can easily produce others.
- Suppose, say, that p divides a_0 but none of the other a_i. We can replace X_i by pX_i for $i = 1, 2, 3$ and then remove the resulting power of p, giving a new surface isomorphic (over $\mathbb{Q}$) to X.

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- In this way we obtain up to four different models. It is not difficult to show that any point in $X(\mathbb{Q}_p)$ reduces to a smooth point modulo p in at least one of these models.

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- In fact, this can be accomplished for any smooth variety over $\mathbb{Q}_p$; such a model is called a weak Néron model.

Unramified places

Theorem

Let X be a smooth, geometrically irreducible variety over k_v. Let $\mathcal{A} \in \operatorname{Br}_1 X$ be an Azumaya algebra split by an unramified extension of k_v. Let $\mathcal{X}/\mathcal{O}_v$ be a smooth model of X, with Z an irreducible component of the special fibre. Then $\operatorname{inv}_v \mathcal{A}(P)$ is constant on the set of points P reducing to Z.

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- This is because the inertia group, by definition, acts trivially on the reduction of X modulo p. So each of the 48 lines on X must be taken to a line with the same reduction modulo p.
- But the 48 lines all have distinct reductions – after all, the reduction of X is a smooth diagonal quartic surface, so contains 48 straight lines.

Tamely ramified places

- Now suppose that $\mathcal{A}$ is split by a totally, tamely ramified Galois extension l/k_v of degree n. There are isomorphisms

$$\mathrm{Br}(l/k_v) \cong k_v^\times / Nl^\times \cong \mathcal{O}_v / N\mathcal{O}_l^\times \cong \mathbb{F}^\times / (\mathbb{F}^\times)^n$$

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- This tells us that, if we have a 2-cocycle describing an element of $\mathrm{Br}(l/k_v)$, and if it takes unit values, then its class is determined by its reduction modulo v.
- With a little work, we can deduce that $\mathrm{inv}_v \mathcal{A}(P)$ only depends on the residue class of P. In fact, we can say more. . .

Tamely ramified places

Theorem

Let X be a smooth, geometrically irreducible variety over k_v, and let $\mathcal{A} \in \operatorname{Br}_1 X$ be an Azumaya algebra split by a tamely ramified Galois extension l/k_v of degree n. Let $\mathcal{X}/\mathcal{O}_v$ be a smooth model of X, with Z a geometrically irreducible component of the special fibre. Then, after possibly modifying $\mathcal{A}$ by a constant algebra, there is a Z-torsor T under $\boldsymbol{\mu}_n$ such that the following diagram commutes.

$$\begin{array}{ccc} X(k_v)_Z & \xrightarrow{\mathcal{A}} & \operatorname{Br} l/k_v \\ \downarrow & & \downarrow \cong \\ Z(\mathbb{F}) & \xrightarrow[T]{} & \mathbb{F}^\times/(\mathbb{F}^\times)^n \end{array}$$

Consequences for diagonal quartics

- On a diagonal quartic surface X, the 48 lines are all defined over some 2-power degree extension of the base field; so this extension is either unramified or tamely ramified except at 2.

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- If X has good reduction, then the reduction is again a smooth quartic surface, so the only torsors under $\boldsymbol{\mu}_n$ are constant; we see again that the Brauer–Manin obstruction there is constant.
- If the reduction of X is a cone, then consider a straight line L in that cone. There are no non-constant torsors under $\boldsymbol{\mu}_n$ on L, even after removing the vertex; so we deduce that the Brauer–Manin evaluation map is constant on the set of points of $X(\mathbb{Q}_p)$ reducing to points on L.